Believe it or Not: The Role of Investor Beliefs for Private Equity Valuation*

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Abstract

I show that investors' beliefs play an important role in the valuation of private equity (PE) funds. The value of PE, in accordance with finance principles, is determined by the expected cash flows discounted for time and risk. Therefore, pricing errors may stem from either an incorrect stochastic discount factor (SDF) or a discrepancy between investors' beliefs and the true distribution of cash flows. I propose an estimation method to back out investors' beliefs from funds' cash flows. I validate recovered beliefs using investors' sentiment surveys. I find that investors' over-optimism about the public market and pessimism about the PE industry, rather than SDF misspecification, offers a potential explanation for the outperformance of PE funds.

Keywords: Subjective Beliefs; Investors' Expectations; Survey Data; Empirical Likelihood; Private Equity Funds.

JEL Codes: G12, G23, G41.

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1 Introduction

Private Equity (PE) refers to investments in non-frequently traded assets. In recent decades, the popularity of PE has grown among institutional investors, particularly in buyout funds (BO). Conventionally, PE is deemed investments in Venture Capital, Buyout, Private Debt, Real Estate, Infrastructure and Natural Resources funds. According to data from Preqin[1], as of 2019, PE funds managed approximately $4.72 trillion in assets, with BO funds accounting for more than $2.7 trillion. In comparison, the U.S. money management industry was estimated to have around $45 trillion in assets in 2019.[2]

The valuation of PE funds is a challenging task due to the non-periodic nature of their cash flows and the lack of return time series data. Conventional valuation methods such as Internal Rate of Return (IRR) and various multiples fail to provide a risk adjustment in line with finance principles (LeRoy (1973)) that is, they do not account for the higher value of payoffs in bad states of the economy. To address these challenges, Kaplan and Schoar (2005) introduced the Public Market Equivalent (PME) metric, which has since become a standard measure for PE performance evaluation in the literature. Building on this, Korteweg and Nagel (2016) further generalized the PME metric by applying the stochastic discount factor (SDF) method to PE valuation.

In this paper, I investigate the role of investors' beliefs for the valuation of PE funds, with a specific focus on Buyout (BO) funds. Asset prices are a result of the interaction between investors' risk preferences, as embodied in the SDF, and their beliefs about the distribution of future cash flows. Thus, pricing errors can arise either from a misspecified SDF or from the incorrect probability distribution used to compute the expectation of future cash flows. I examine how investors' beliefs can deviate from the true probability distribution for various investors' risk preferences, which are represented by several commonly used SDFs, and how these deviations affect the prices in PE.

I estimate investors' subjective beliefs from PE cash flow data using a two-stage estimation procedure. First, I estimate the SDF parameters by pricing the public market cash flows, following the approach of Korteweg and Nagel (2016). Then, I apply the estimated SDF to price the excess cash flows, which are calculated as the difference between the aggregate PE fund cash flows and the aggregate cash flows of a mimicking fund that invests in a stock market index.[3]

In the second step, I determine an alternative probability distribution that eliminates the pricing errors from the first step. This is achieved by minimising the divergence between this alternative distribution and the true distribution, subject to the constraint that the expected

[1] Leading provider of data, analytics, and insights for PE funds. [2] In accordance with SEC Asset Management Advisory Committee - 16th September 2020. [3] more details in 2.3.1 section

value of the discounted excess cash flows is equal to zero:

$$\min_{\hat{\hat{P}}} D(\hat{\hat{P}} \,||\, \hat{P}) \text{ subject to } E^{\hat{\hat{P}}}\left[M_{1;t} \sum_{i=1}^{N} CF^{exc}_{i;t} \right] = 0;$$

where $D(\hat{\hat{P}} \,||\, \hat{P})$ is the Kullback-Leibler divergence of $\hat{\hat{P}}$, the alternative probability distribution of discounted excess cash flows with $M_{1;t}$, from $\hat{P}$, the empirical probability distribution. Investors' estimated beliefs correspond to the probability distribution $\hat{\hat{P}}$.

First, I find that investors' beliefs help explain PE valuation. In particular, I find that abnormal performance among PE funds can be linked to investors demanding an excessive premium for holding PE investments. On average, investors tend to be overly optimistic about the expected performance of the market portfolio while being pessimistic about PE funds' ability to generate cash flows.

The average excess cash flows are perceived by investors as smaller than what has been historically observed. To support this conclusion, I compare the observed and belief-adjusted distributions of average excess cash flows from 1996 Q1 to 2014 Q4, as depicted in Figure 1. To estimate market beliefs, I use the SDF implied by a power-utility Capital Asset Pricing Model (CAPM). As shown in Figure 1, the belief-adjusted average excess cash flows exhibit a thicker negative tail and a more left-skewed distribution compared to the historical cash flows.

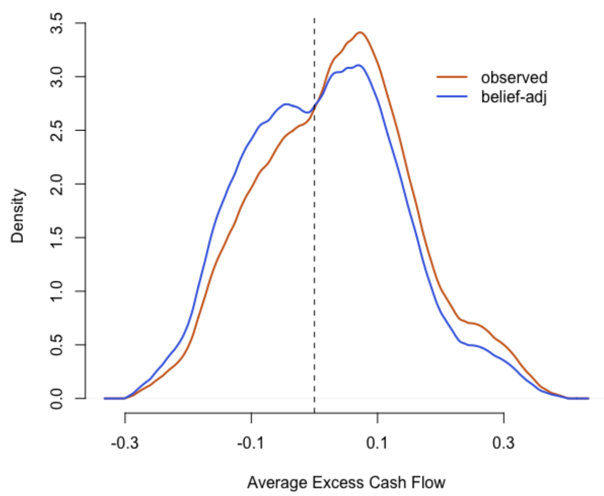

Figure 1: Average Excess Cash Flow: Original vs. Belief-Adjusted. The figure plots PE funds' observed and belief-adjusted average cash flows for 1996 Q1 – 2014 Q4. The distribution for belief-adjusted cash flows is based on model-implied (CAPM) subjective beliefs. The excess cash flows are in $million. Epanechnikov kernel estimates of the densities are used.

Second, I demonstrate the validity of the estimated beliefs by showing that they correspond to actual investor sentiment as documented in survey data. I use survey data for three groups of investors: individual stock market investors, institutional stock market investors, and private equity investors. To assess sentiment among stock market investors, I analyse data from the Gallup Investor Survey, the Economic Sentiment Indicator (ESI) from Eurostat, and Robert Shiller's Investor Survey for both individual and institutional investors. For private equity investors, I examine the Silicon Valley VC Confidence Index (SVVCCI) and the Central Europe PE Confidence Index (CEPECI). The SVVCCI is a quarterly survey of venture capitalists in the San Francisco Bay Area, conducted by Cannice and Goldberg (2009), while the CEPECI is a semi-annual survey of private equity professionals in Central Europe, conducted by Deloitte. The results show a positive correlation between the PE sentiment indices and the belief-adjusted average excess cash flows, indicating that the estimated beliefs contain information about investors' expectations.

The results discussed earlier hold up well against various modifications to the underlying economic models. To further demonstrate this, I also extract beliefs under different models, including the Consumption-CAPM (C-CAPM) by Rubinstein (1976), the External Habit model by Campbell and Cochrane (1999), and the Long-Run Risks model based on Epstein-Zin preferences by Bansal and Yaron (2004). These models provide alternative ways to describe the risk preferences and structure of the economy, and the robustness of the results to these modifications underscores the validity of the findings.

Third, I examine the potential impact of mis-specification of the SDF on estimated beliefs. To do this, I orthogonalize the recovered beliefs by considering public market factors that could affect risk preferences. I consider equity risk premium (as discussed by Haddad, Loualiche, and Plosser (2017)), the Moody's BBB to AAA credit spread, the Chicago Board Options Exchange S&P 100 Volatility Index, and the aggregate liquidity factor (as described by Pastor and Stambaugh (2003)). The results show that the orthogonalized beliefs are highly correlated with the original beliefs, and this correlation is even stronger for beliefs estimated using SDFs that produce a lower pricing error for excess cash flows. This suggests that a more accurate SDF produces more informative beliefs.

Thus, the results suggest that estimated beliefs might reflect omitted risk factors in the SDF. One such factor that may be missing in investors' risk preferences is market volatility. While the impact of public market volatility on the venture capital industry has been studied by Gompers, Kovner, Lerner, and Scharfstein (2008), who found that investors increase their investments in VC during times of heightened volatility, there has been limited research on the effect of volatility on PE funds. However, the findings of this paper suggest that volatility may have a similar impact on PE funds as it does on the VC industry.

The remainder of the paper is organised as follows. In Section 2, I present the methodology and the moment condition that I use to estimate the investors' beliefs from the observed excess cash flows. Section 3 details the results of the model-implied subjective beliefs and provides an explanation for the existence of positive excess cash ows. Additionally, I explore various risk preferences assumptions and conduct a bootstrap exercise to compare the economic implications of the estimated beliefs in Section 3. Section 4 compares the estimated beliefs with survey data from individual, institutional, and PE investors. Finally, the paper concludes with suggestions for future research.

Related Literature. The role of investor beliefs in shaping asset prices has been the subject of growing interest in the literature. The idea that asset prices re ect information about risk and uncertainty has deep roots in economics, dating back to the work of Knight (1921), who made the intuitive distinction between risk and uncertainty. Miller (1977) was one of the rst to show that investors' perceptions of uncertainty can distort asset prices. In his framework of a simple supply-demand model with short-selling constraints, he demonstrates that seemingly high security prices can result from substantial divergences of opinion about the future returns of securities. In this sense, Miller (1977) can be seen as a precursor to the idea that investors' subjective beliefs and their heterogeneity are important factors in asset pricing. For a long time, the nancial literature was dominated by the rational expectations paradigm, as proposed by Muth (1961). This approach equates the beliefs held by investors with the true (objective) data generating distribution, e ectively excluding any potential impact of investors' opinions on prices and eliminating the need to estimate their beliefs.

Stefan Nagel, in his discussion on the future of asset pricing in Brunnermeier et al. (2021), recognises the signi cance of rational expectations (RE) based approaches, as the rst approximation of investors' beliefs. However, he also acknowledges the limitations and fallacies associated with this paradigm. To overcome these limitations, it is necessary to incorporate investors' expectations into asset pricing models, allowing for their beliefs to di er from the true probability distribution. Behavioral biases can lead to deviations from rationality, as discussed by Barberis and Thaler (2003). To validate and potentially relax the RE assumption, researchers can collect survey data to elicit investors' subjective probabilities about economic choice-related problems, as suggested by Manski (2004).

Greenwood and Shleifer (2014) is the rst work which explicitly compares investors' survey expectations with expected returns predicted by the rational expectations (RE) model. Their ndings, which showed a strong negative correlation between the two, challenged the validity of the RE paradigm. This led to the development of the theoretical approaches for recovering investors' subjective beliefs from asset prices, as seen in the works of Ross (2015) and Chen, Hansen, and Hansen (2020). These works have been followed by a growing body of empirical studies aimed at uncovering investors' beliefs. Ghosh and Roussellet (2020) recover price-consistent conditional beliefs for a representative investor. Their paper addresses the concern that the recovered beliefs might be just a product of preferences misspeci cation. Ghosh, Korteweg, and Xu (2020) focus on recovering beliefs of heterogeneous investors and conduct a concise analysis of recovered beliefs. For an overview of the literature on investors' beliefs, one may refer to Adam and Nagel (2022). In this paper, I add to this literature by extracting the unconditional beliefs of investors for private equity (PE) funds.

The paper also contributes to the second strand of the literature that employs Empirical Likelihood-based methods in addressing economic questions. The Empirical Likelihood (EL) estimator was rst proposed by Owen (1988). Despite its appealing theoretical properties, as highlighted by Anatolyev and Gospodinov (2011), the structure of EL objective function and rst-order conditions can raise computational challenges. In addition, the model misspeci cation problems that are prevalent in the asset pricing literature may lead to the EL estimator numerically ill-behaving. To address these challenges, Smith (1997) proposed the Generalised Empirical Likelihood (GEL) class of estimators. This paper utilises the exponential tilting (ET) estimator, a special case of the GEL estimator. I focus on selecting the probability measure that has the least discrepancy from the true probability distribution, while also satisfying the moment condition for excess cash ows of PE funds.

The use of Empirical Likelihood (EL)-type methods has gained popularity in the nancial literature. Stutzer (1996) were one of the rst to use EL-type methods to extract the risk-neutral probability distribution. Julliard and Ghosh (2012) used EL-type methods to examine the empirical validity of the rare disasters hypothesis in the context of the equity premium puzzle. Almeida, Ardison, and Garcia (2020) applied EL-type methods to evaluate the performance of hedge funds by utilizing the information contained in pricing errors to construct a non-parametric SDF. Ghosh, Julliard, and Taylor (2017) used EL-type methods to estimate the unobservable component of the SDF, construct entropy bounds on candidate SDFs, and propose the minimum necessary adjustments to correctly price asset returns. Other notable applications of EL-type methods in the asset pricing literature include works by Almeida and Garcia (2012, 2017), Post, Karabati, and Arvanitis (2018), and Almeida, Ardison, and Garcia (2020).

Moreover, I contribute to the rich empirical literature on evaluating the performance of PE funds, building on the works of authors such as Kaplan and Schoar (2005), Korteweg and Nagel (2016), Gupta and Van Nieuwerburgh (2021), Gredil, Sorensen, and Waller (2019), Ang, Chen, Goetzmann, and Phalippou (2018), Harris, Jenkinson, and Kaplan (2014), Robinson and Sensoy (2016), and many others. I show that subjective beliefs can substantially improve existing models by eliminating pricing errors.

2 Methodology

2.1 SDF Pricing

The classical asset pricing theory defines the value, V_t, of an asset at time t as the adjusted value for time and risk using a stochastic discount factor (SDF), $M_{t;t+1}$. The value of the asset depends on the expected present value of the future cash flow, CF_{t+1}, and the residual value, V_{t+1}, at time $t+1$:

$$V_t = E_t^P[M_{t;t+1}(CF_{t+1} + V_{t+1})] \tag{1}$$

The value of an asset in classical asset pricing theory is not only determined by the risks associated with the investment, but also by investors' beliefs about the distribution of future cash flows. The Euler equation 1 consists of three elements: (1) the stochastic future cash flows ($CF_{t+1}; V_{t+1}$), (2) the one-step-ahead SDF ($M_{t;t+1}$), and (3) the true probability measure (P). The valuation error can stem from either an incorrect reflection of risk preferences in the SDF or a failure to accurately capture the representative investor's knowledge of the data-generating process through the rational expectations (RE) assumption.

This paper focuses on the violation of the rational expectations (RE) hypothesis as the primary cause of valuation errors. Although the use of REs is widely accepted for analytical purposes, the paper proposes an alternative formulation of the Euler equation that incorporates investors' beliefs, represented by the measure $\tilde{P}$, rather than the traditional P measure. The equation is shown below:

$$V_t = E_t^{\tilde{P}}[M_{t;t+1}(CF_{t+1} + V_{t+1})] \tag{2}$$

This paper concentrates on the impact of the discrepancy between the true market conditions (P) and the beliefs held by investors ($\tilde{P}$) on the valuation of PE funds, and shows that the standard SDF approach is not adequate in explaining the abnormal performance of PE funds. The evidence supporting this approach is based on the literature that challenges the RE hypothesis. For example, Greenwood and Shleifer (2014) demonstrates that while expected returns implied by models are countercyclical, investors' expectations as revealed through survey data are procyclical. Additionally, Ghosh, Korteweg, and Xu (2020) provide evidence for the presence of multiple types of investors in the market with heterogeneous beliefs.

I assess the performance of PE funds using the Preqin data and the Generalized Public Market

Equivalent (GPME) metric introduced by Korteweg and Nagel (2016). The GPME approach calculates the Net Present Value (NPV) of PE cash flows using the SDF as a discount factor. I find the abnormal performance for PE funds of 25 cents per one invested dollar.[4]

$$NPV_i^{PE} = \sum_{t=1}^{T} CF_{i;t}^{PE} M_{1;t} \tag{3}$$

Here $t = 1; \ldots; T$ is the span of all dates for which I observe PE cash flows and $M_{1;t}$ is the multi-period SDF that compounds the single period discount factors:

$$M_{1;1} = 1; M_{1;2} = M_{1;2}; M_{1;3} = M_{1;2}M_{1;3}; M_{1;t} = M_{1;2}M_{1;3} \ldots M_{1;t}$$

Under the null hypothesis, the NPV of a PE fund is expected to be zero, i.e. $E[NPV_i] = 0$. If this holds true for each fund, then the sum of NPVs across all funds should also be zero:

$$\sum_{i=1}^{N} E_t^P \sum_{t=1}^{T} CF_{i;t}^{PE} M_{1;t} = 0 \rightarrow E_t^P \sum_{i=1}^{N} \sum_{t=1}^{T} CF_{i;t}^{PE} M_{1;t} = 0$$

Similarly, a benchmarking fund, which invests in an aggregate market index such as CRSP, has an expected NPV of zero:

$$E_t^P \sum_{i=1}^{N} \sum_{t=1}^{T} CF_{i;t}^{mrkt} M_{1;t} = 0$$

The only source of abnormal performance in this case is the excess cash flows, defined as the difference between the cash flows of the PE fund and the benchmarking fund:

$$E_t^P \sum_{t=1}^{T} M_{1;t} \sum_{i=1}^{N} [CF_{i;t}^{PE} - CF_{i;t}^{mrkt}] = E_t^P \sum_{t=1}^{T} M_{1;t} \sum_{i=1}^{N} CF_{i;t}^{exc} = 0 \tag{4}$$

Equation 4 implies that the NPV of both the aggregate private equity and market funds are equal to zero. This result is a direct consequence of utilising the GPME as the performance metric. However, it can be argued that a stronger condition applies to the excess cash flows: these should be equal to zero at each individual period. This condition stems from the existence of the SDF, which is a crucial element in the GPME performance metric. The condition implies that the representative agent should be indifferent at each period in regards to choosing which asset to invest in, thus resulting in the excess cash flows being equal to zero at each period. To estimate beliefs, the following moment condition is used:

[4] For the estimation details, see Table 4

$$E_t^{P} M_{1;t} \sum_{i=1}^{N} CF_{i;t}^{exc} = 0 \tag{5}$$

The presence of abnormal performance for private equity funds, under certain parameters, suggests that Equation 5 is not satisfied given the combination of risk preferences ($M_{1;t}$) and probability distribution (P). In order to account for this discrepancy, this paper introduces the concept of subjective beliefs (P), which may differ from the true probability distribution (P) implied by the RE hypothesis. To extract these subjective beliefs, I employ a two-stage procedure. In the first stage, the parameters of the SDF are estimated under the empirical probability distribution ($\hat{P}$) using the moment condition provided by the GPME framework, assuming that the benchmark fund invested in the public market is perfectly priced. In the second stage, the estimated SDF parameters are applied to the excess cash flow, and the resulting pricing errors are used to identify an alternative probability distribution ($\hat{P}$) that is closest to ($\hat{P}$) in the Kullback-Leibler sense and satisfies the moment condition with zero pricing error.

2.2 Empirical Likelihood Estimation

I employ an empirical likelihood (EL)-based methodology, namely the exponential tilting (ET) estimator (as introduced by Efron (1981), Kitamura and Stutzer (1997), and Imbens, Spady, and Johnson (1998)), to estimate the subjective beliefs. To back out beliefs I use the moment condition for excess average cash flows presented in Equation 5. I adopt an information-theoretic approach based on the Kullback-Leibler (KL) divergence to extract a probability distribution that satisfies the moment condition without considering the parameter estimation problem. The probability distribution closest to the true one is interpreted as the investor's subjective beliefs.

Let us now formalise the problem. For each SDF parameter $\theta \in \Theta$; $\theta = (a; b)$, I define an alternative probability distribution P as follows:

$$E^{P} M_{1;t}(\theta) \sum_{i=1}^{N} CF_{i;t}^{exc} = 0 \tag{6}$$

The objective is to find a particular distribution P that is closest to P in a statistical sense. This can be achieved by solving the following convex optimisation problem:

$$\min_{P} D(P \| P) \quad \min_{P} \int \log\left(\frac{dP}{dP}\right) dP \quad \text{subject to } E^{P} M_{1;t}(\theta) \sum_{i=1}^{N} CF_{i;t}^{exc} = 0 \tag{7}$$

$D(P\,\|P)$ is a measure of the closeness between the subjective probability distribution P and the true probability distribution P. The KL divergence is non-negative and is equal to 0 if and only if P is equal to P, which is the case under the rational expectations (RE) hypothesis.

To find the solution to this discrepancy minimisation problem, consider the following objective:

$$\min_{p_t} D(\hat{P}\,\|\hat{P}) \quad \min_{p_t} \sum_{t=1}^{T} \frac{1}{T} \log \frac{1}{T p_t}$$

where $1/T$ denotes the empirical probability distribution, and p_t is the discrete distribution of subjective beliefs that assigns probabilities to the same points in time as $CF_{i;t}$; $t = 1; \ldots; T$. The solution to this problem can be found in Csiszar (1975) and is given by:

$$p_t = \frac{e^{(\ )'[M_{1;t}(\ ) \sum_{i=1}^{N} CF_{i;t}^{exc}]}}{\sum_{t=1}^{T} e^{(\ )'[M_{1;t}(\ ) \sum_{i=1}^{N} CF_{i;t}^{exc}]}} \tag{8}$$

where $(\ )$ is the solution to the unconstrained convex problem:

$$\min E_t^P e^{(\ )'[M_{1;t}(\ ) \sum_{i=1}^{N} CF_{i;t}^{exc}]}$$

$(\ )$ is also the Lagrange multiplier for the empirical likelihood problem:

$$\max_{p_1;\ldots;p_T} \sum_{t=1}^{T} \log p_t \quad \text{s.,t.} \quad p_t M_{1;t}(\ ) \sum_{i=1}^{N} CF_{i;t}^{exc} = 0; \quad \sum_{t=1}^{T} p_t = 1: \tag{9}$$

The subjective beliefs extracted using an information-theoretic approach and EL-setup with an ET estimator are equivalent. The ET estimator searches for the probabilities that maximise the non-parametric log-likelihood of the observed data, subject to the constraint that the unconditional Euler equation is satisfied. Therefore, by solving problem (9), I can estimate the distribution of investors' beliefs that would make zero pricing error for excess cash flows moment condition.

2.3 Data

In this sub-section, I will describe the cash flow data of PE funds that employ a BO strategy. The data spans the period from 1980 to 2020 and was sourced from Preqin. For the purpose of my analysis, I have chosen a sub-sample of PE funds that were incepted between 1980 and 2015.

The cash flows data consist of three elements: capital calls, capital distributions, and valuations. Capital calls are when the General Partner (GP) requests capital from the Limited Partner (LP) in order to make investments. Capital distributions, on the other hand, occur when the LP begins to receive returns from past investments. Finally, valuations provide information on the remaining capital to be paid back to investors.

However, valuations can sometimes be distorted and have been known to have potential issues, as reported by Phalippou and Gottschalg (2009). This is the main reason why only funds with vintages before 2015 are included in the analysis. Additionally, the sample is limited to closed and liquidated funds. Liquidated funds are those that have terminated their operations and distributed all capital back to investors, while closed funds are those that have completed their fundraising campaign but have not yet distributed all capital.

2.3.1 Average Excess Cash Flows. To estimate the subjective beliefs, I construct the average excess cash flows ($CF^{exc}_{i;t}$). The analysis focuses on quarterly excess cash flows from 1996 Q3 to 2014 Q4. In this article, I use the average excess cash flows instead of aggregated data, and limit the sample due to concerns about stationarity. The EL methodology requires stationary time series as the underlying moment condition for valid statistical inference. For further information, please refer to Appendix A.

I construct the aggregate excess cash flows by subtracting the aggregate public market benchmarking cash flows ($\sum_{i=1}^{N} CF^{mrkt}_{i;t}$) from the aggregate PE cash flows ($\sum_{i=1}^{N} CF_{i;t}$). The market benchmarking fund is created following the methodology described in Korteweg and Nagel (2016). Essentially, the benchmarking fund receives the same investments as individual PE funds, and then invests in the CRSP value-weighted index, while maintaining the timing of capital calls and capital distributions unchanged.

Figure 2 plots the average excess cash flow for PE funds from 1996 Q1 to 2014 Q4. The range of the average excess cash flow fluctuates from $0.21 billion to $0.31 billion. Meanwhile, the aggregate excess cash flow fluctuates from $55 billion to $75 billion. The mean of the depicted average excess cash flows is $0.04 billion. It is important to note that all cash flows are adjusted for inflation and presented in 1990 dollars. To assess the economic significance of the excess cash flows, I compare them to the average payout, which I define as the sum of capital distributions across funds in each period. The mean of this ratio is 6.6%, which represents a substantial portion of the average PE cash flow.

2.3.2 Summary Statistics. Table 1 presents the descriptive statistics for the sample of private equity (PE) funds used in the analysis. The sample consists of 976 funds raised by 454 PE firms. The average PE firm in the sample raised two funds, with an average commitment of $1848 million. There is a long right tail in the distribution of fund sizes, primarily due to older PE firms that have

Figure 2: Average Excess Cash Flows and Its Components. The figure plots the average excess cash flow for PE funds (black line) for 1996 Q1 – 2014 Q4. The average excess cash flow is the difference between average PE cash flows (brownish line) and average CRSP cash flows (blue line). All numbers are in $ millions.

raised more than one fund, as these firms tend to raise increasingly larger funds over time. The average fund in the sample has 30 unique cash flows (including capital calls and distributions) and the time between its first and last observed cash flow is 10 years. The median (average) fund has an internal rate of return (IRR) of 10.78% (11.34%) and a total value to paid-in capital (TVPI) ratio of 1.52 (1.65). The GMPE metric is calculated based on the GMM estimation of risk preferences entering the SDF, as presented in Korteweg and Nagel (2016). The results of the GMM estimation are displayed in Table 4. The abnormal performance for the PE funds in the sample is positive and equal to 25 cents and 15 cents for the GMPE and PME metrics, respectively.

3 Empirical Results

3.1 Subjective beliefs of PE investor

I first illustrate my methodology for the CAPM SDF used in Korteweg and Nagel (2016). For this specification of risk preferences, the one-period SDF takes the form $M_{t;t+1} = \exp(a - b r_{m;t+1})$, where a governs the unconditional mean of the log SDF, and b is the coefficient of relative risk aversion. I estimate the SDF parameters via exact pricing the aggregate public market represented by CRSP index and Treasury Bill benchmarks.[5]

The discounted excess cash flows $M_{1;t}(\frac{1}{N_t}\sum_{i=1}^{N} CF^{exc}_{i;t})$ are presented in Table 2. For comparison,

[5] In this section, I use the following estimates: $\hat{a} = 0.16$ and $\hat{b} = 3.63$. See the estimation results in Table 4.

Table 1: Descriptive Statistics: PE Fund Data. Source: Preqin. Only closed and liquidated funds with a commitment of at least \$5 million in 1990 dollars incepted in 1980-2015 are considered. Fund size is the total commitment to the fund, in millions of dollars. Fund effective years is the time between the first and last observed cash flows of a fund. TVPI is defined as the sum of cash distributions plus any net asset value remaining in the fund divided by the sum of capital takedowns. GPME (Generalized Public Market Equivalent) is average discounted cash flows across funds with SDF of the form: $M_{1;h} = \exp(ah + b\sum_{j=1}^{h} r_{t+j})$, where r_{t+j} is log market returns. The special case with $a = 0$, $b = 1$ corresponds to the PME.

	Mean	Median	St.dev
# funds	976		
PE firms	454		
Funds/PE Firms	2.15	2.00	1.64
Fund size	1848	783	2812
Fund effective years	10.31	10.63	4.60
# of cash flows / fund	29.37	30.00	12.29
IRR, %	11.34	10.78	13.96
TVPI	1.65	1.52	0.76
PME, % – a/r per \$1 com.	15.43	11.85	39.64
GPME, % – a/r per \$1 com.	25.43	-0.71	125.38

the table also reveals quarterly average excess cash flow without risk adjustment. The discounted excess cash flows are close to zero, though the pricing error is not zero. The aim of this paper is to extract beliefs consistent with the observed excess cash flows that would eliminate the pricing error induced by SDF misspecification. I find the alternative probability distribution using Equation 8; thus, it guarantees: $p_t M_{1;t} \sum_{i=1}^{N} CF_{i;t}^{exc} = 0$, where p_t are subjective beliefs of PE investor about average excess cash flows.

Table 2: Quarterly Average Excess Cash Flows. The table presents observed and discounted (using CAPM model) excess cash flows. The statistics are presented in \$ millions for 1996 Q1 – 2014 Q4.

	Min	Q1	Median	Mean	Q3	Max
Discounted	-0.166	-0.016	0.014	0.016	0.037	0.215
Observed	-0.207	-0.048	0.038	0.036	0.107	0.308

The results for model-implied beliefs and belief-adjusted components of excess cash flows are presented in Figure 3. The upper panel reveals the profile of subjective beliefs for CAPM SDF. Investors' opinions are significantly affected by the recession periods caused by the dot-com bubble and the global financial crisis (shaded areas). The spikes around 2003 Q2 year and 2009 Q3 year signify that the investors adjust their beliefs gradually, not immediately after crisis events, but the speed of adjustments increased between these two disastrous events.

The general interpretation of model-implied beliefs might be the following. The dashed line on the graphs corresponds to the empirical (historical, $=1$) probability of average excess cash flow occurrence. Consequently, the deviation from this level signifies that investors' expectations were discrepant with the market. For example, the periods after severe recessions demonstrate the substantial deviation of subjective beliefs from beliefs implied by the RE hypothesis. After such periods, the highest pricing error is observed. Therefore, the crisis episodes strikingly demonstrate the importance of the subjective beliefs approach for PE valuation when the asset pricing model under the empirical probability distribution cannot explain the market movement.

To understand how investors make investment decisions about PE funds, I inspect belief-adjusted components of excess cash flow - the PE cash flow and the aggregate public market cash flow. The middle and bottom panels of Figure 3 present the economic story behind the subjective beliefs. Here I address the question of why positive abnormal performance exists in PE. The potential reason is that the investors are overpessimistic about PE cash flows and, at the same time, overoptimistic about the public market. The middle panel shows the difference between belief-adjusted and observed aggregate cash flows, what I call **belief gap**. At approximately 63% time of the observed period, the belief gap is negative; it means investors usually tend to underestimate the performance of PE funds. The pessimism is especially salient around crises. However, the behavioural pattern is not trivial: at the beginning of the global financial crisis (2008 Q1 – 2009 Q1), investors severely underestimated its impact on cash flows generated by PE funds. Later, the optimism was replaced by panic; in 2009 Q3, the aggregate difference between expected and realised PE cash flow was around $25 million (the largest one in the analysed sample).

The bottom panel conducts the same analysis for the public market. The picture is diametrically opposite; at approximately 65% time of the observed period, the belief gap is positive and larger in amplitude. Investors tend to overestimate public market cash flow even around disastrous events. Only in 2009 Q2 did they estimate the CRSP cash flow to be $6 million less than it occurred. Nonetheless, at the peak of the crisis (2008 Q4), PE investors strongly believed that the public market was a good hedge asset, what transformed into a $32 million overestimation of public market cash flow.

Mainly, results correspond with strong beliefs in the public market's ability to provide them with a good hedge opportunity, especially during the recession periods. Such beliefs force them to demand excessive compensation for the risk of holding PE funds' stakes, which leads to the observed positive abnormal performance of PE funds. Overconfidence (e.g. about the current portfolio holdings) is a well-known phenomenon documented in the behavioural finance literature, see Glaser and Weber (2007). The behavioural biases might be attributed not only to unsophisticated investors. Elan and Goodrich (2010) show that even experts often overestimate their abilities

Figure 3: Subjective Beliefs. On the upper panel, the figure plots the model-implied probabilities obtained using an Empirical Likelihood – type estimator. These probabilities are estimated for 1996 Q1 – 2014 Q4. On the middle and bottom panel, the figure plots the difference between belief-adjusted and observed aggregate cash flows (Belief Gap) for PE funds and CRSP mimicking funds (in $million). Shaded areas are NBER recession periods.

to forecast the future of their investments more than the average person does.

3.2 Extracting Beliefs for Different SDFs

My methodology allows extracting beliefs under different assumptions on the risk preferences and structure of the economy. I can use models with parameters estimated on the cash flow data and off-the-shelf parameters obtained from the literature. In this sub-section, I show that the inference about subjective beliefs remains similar to section 3.1, when applying the method to a broader class of consumption-based asset pricing models. The reasons for this exercise might vary: from checking robustness to comparing counterfactual implications of beliefs estimated under different assumptions about risk preferences. For example, Gredil, Sorensen, and Waller (2019) highlight that the largest investors in PE are pension funds and endowments, who try to hedge the consumption risks rather than target the excess returns over the public market benchmark. The inter-generational redistribution of income dictates such investment targets. Therefore, they are ready to bear extra liquidity costs by locking up the capital for more than ten years. Cejnek, Franz, and Stoughton (2017) state that the long-run risks and the external habit models match the risk preferences of the PE investors. Here I test the aforementioned models to see if they can better explain PE valuation and how beliefs extracted under assumptions underlying these models differ from the baseline.

3.2.1 Mainstream SDFs Construction and Estimation.

Additionally to exponential CAPM, I consider the C-CAPM (Consumption-CAPM) of Rubinstein (1976), the EHB (external habit formation) model of Campbell and Cochrane (1999), the LRR (long-run risks) model of Bansal and Yaron (2004). Also, I extend the exponential CAPM of Korteweg and Nagel (2016) with the size premium of Fama and French (1992) (CAPM + SMB) and consider Kaplan and Schoar (2005) version of CAPM SDF (K-N CAPM). In this sub-section, I explain how I construct and estimate these models using GMM (Generalised Method of Moments).

Investors in Campbell and Cochrane (1999) model with habit formation want to preserve their consumption above the habit level existing in the economy. This behaviour matches the fact that the main PE investors (who are institutional) seek to smooth future consumption. The consumption in the economy of Bansal and Yaron (2004) is assumed to have a small but persistent component that can influence future returns. Mainly, the long-run risks investor is averse to future uncertainty about the long-term consumption and also seeks to smooth it. The idea is very similar to Campbell and Cochrane (1999), but the mechanism is slightly different, and generally speaking, asset pricing implications for the LRR model differ substantially from EHB. These two models successfully explain the equity premium and the risk-free rate puzzles with reasonable calibrating parameters. I test them to understand if they can give a di erent economic intuition for the PE valuation process. I use C-CAPM and CAPM with SMB as benchmarking models since they are relatively more simplistic in assumptions and easier to bring to the data.

Table 3: SDFs Construction . The table presents the functional form of SDFs used in the analysis (Panel A). The estimation of risk-preferences parameters is conducted separately from the estimation of subjective beliefs or does not occur at all. I use 'hats' only for estimated parameters, for instance $\hat{a}, \hat{b}$. For long run risk and external habit models (LRR and EHB) to construct the one-period-ahead SDF, it needs to specify the consumption-based time-series and calibration parameters (Panel B). The notation is the following (all lowercase variables in log): b { RRA, $r_{m;t+1}$ { aggregate market returns, $r_{smb;t+1}$ { returns on the size premium, c_t { consumption, h_t { external habit de ned as in Campbell and Cochrane (1999), x_t { long run risks component of consumption growth de ned as in Bansal and Yaron (2004), the rest variables are calibration parameters.

Panel A: Functional form		
Model	One-period-ahead SDF (log)	Paper
K-N CAPM	$m_{t+1} = \hat{a} \quad \hat{b} r_{m;t+1}$	Korteweg & Nagel (2016)
K-S CAPM	$m_{t+1} = \quad r_{m;t+1}$	Kaplan & Schoar (2005)
CAPM+SMB	$m_{t+1} = \hat{a} \quad \hat{b} r_{m;t+1} \quad \hat{c} r_{smb;t+1}$	Korteweg & Nagel (2016)
C-CAPM	$m_{t+1} = \log(\ ) \quad \hat{b} \quad c_t$	Rubinstein (1976)
EHB	$m_{t+1} = \log(\ ) \quad \hat{b} (\quad c_t \quad h_t)$	Campbell & Cochrane (1999)
LRR	$m_{t+1} = \log(\ ) \quad \frac{1}{\ } x_t \quad c \frac{1 \quad \hat{b}}{(1 \quad x \quad c)} e_{x;t+1} \quad \hat{b} e_{c;t+1}$	Bansal & Yaron (2004)

Panel B: Construction	
Model	Consumption-based time series
EHB	$H_t = C_t \quad C_t S_t$
	$s_t = (1 \quad _{QQ}) \log(S) + \quad _{QQ} s_{t \ 1} + \quad (s_{t \ 1}) u_t$
	$(s_t) = 1 = S \ \sqrt{1 \quad 2(s_{t \ 1} \quad \log(S))} \quad 1, \ s_t \quad s_{max}$, and 0 otherwise
	$S_{max} = \exp(\log(S) + \frac{1}{2}(1 \quad S^2))$
	$S = \quad (\quad c_t) \sqrt{\frac{\ }{(1 \quad _{QQ})}}$
	$_{QQ} = \ ^3_{mm} = 0{:}89^3, \quad = 2, \quad = 0{:}998$
LRR	$x_t = \ \hat{} \ c_t, \quad c_t = [\quad c_{t \ 1}; \quad cy_{t \ 1}; pd_{t \ 1}; rf_{t \ 1}; spread_{t \ 1}]$
	$e_{x;t+1} = x_{t+1} \quad _x x_t, \ e_{c;t+1} = \quad \hat{c}_{t+1} \quad x_t$
	$sd(e_{c;t+1}) = \quad _{QQ}, \ sd(e_{x;t+1}) = \quad _e \quad _{QQ}$
	$_x = \ ^3_{mm} = 0{:}987^3, \quad = 2, \quad _c = 0{:}9649, \quad _{QQ} = \ \sqrt{3} \quad _{mm} = \ \sqrt{3} \quad 0{:}0078,$
	$_e = 0{:}1085$ and $= 0{:}998$

Table 3 exhibits the functional form of each SDF. For all models, I estimate the relative risk aversion (RRA) parameter. The estimation procedure is conducted in the spirit of Korteweg and Nagel (2016): SDF parameters are chosen to perfectly price the public equity market. For all estimations, I use exactly identified GMM. For CAPM and CAPM+SMB models, I use additional factors like Treasury Bill and small growth portfolio benchmark to estimate unconditional mean of the log SDF ($\hat{a}$) and price of SMB risk ($\hat{c}$). The special attention deserves the construction of LRR and EHB models due to higher complexity of these models, see Panel B of the table. To construct them, I closely follow the original paper for the EHB model and Colacito and Croce (2011) for the LRR model. I take the consumption time series from the National Income and Product Accounts of the Bureau of Economic Analysis. The consumption-based time series for LRR, EHB and C-CAPM models are constructed from 1949 Q4 to 2019 Q4. Consumption-based asset pricing models are susceptible to the consumption data, which can result in the different relative performances of the PE industry, for more details, see Kroencke (2017).

The estimation of all models is presented in Table 4. The point estimate of RRA coefficient for K-N CAPM model is 3.63, substantially higher than 1, imposed by Kaplan and Schoar (2005) log-utility model. J-test rejects the null hypothesis of zero pricing errors, and GPME is equal to 25 cents of outperformance for each committed dollar. The K-S CAPM column shows that PME estimate across all funds is 0.15. Suppose, the researcher would try the alternative SDFs, for example, as in Rubinstein (1976), where the discount factor explicitly depends on the changes in aggregate consumption. In that case, the results about average PE performance will not alter significantly. The abnormal return for C-CAPM model is 14 cents, which is in the ballpark of two previously discussed models.

The abnormal performance is equal to 10 cents and 13 cents for long-run risks model and external habit model accordingly, LRR and EHB columns of Table 4. These models produce the smallest pricing error, which is by definition abnormal performance. This result confirms the conclusion of Cejnek, Franz, and Stoughton (2017) about the importance of considering EHB and LRR risk preferences as the best models for PE investors' behaviour. All else equal, the model-implied beliefs should be more precise for the pricing kernels with the smallest unexplained performance.

Then I apply SDFs with estimated parameters to the average excess cash flows. Figure 4 visualises the pricing errors for excess cash flows and reports the Mean Standard Errors (MSE) for different SDFs. The EHB and LRR models price excess cash flow with the highest precision among others. This result is not surprising since the excess cash flows are the only source of abnormal performance for the GMPE-like metrics. K-N CAPM, K-S CAPM and C-CAPM SDFs are almost identical in terms of pricing errors. To explain the remaining pricing errors, I introduce subjective

Table 4: Performance Under Different SDFs. I estimate the abnormal performance (GPME) by discounting PE fund cash flows with the SDF from Table 3. K-N CAPM – Korteweg and Nagel (2016) SDF, K-S CAPM – Kaplan and Schoar (2005) SDF, C-CAPM – Rubinstein (1976) SDF, EHB – Campbell and Cochrane (1999) SDF, LRR – Bansal and Yaron (2004) SDF. p-values for t-tests in round brackets. The estimation of SDF parameters is conducted only for parameters for those p-values are provided; for the rest, SDF parameters are taken from the literature. For all models, parameters are chosen to correctly price benchmark funds that receive the same inflows as the PE funds but that invest in the CRSP value-weighted index and T-bills for the sample of funds incepted 1980 – 2015. The bottom panel presents J-test that abnormal performance is equal to zero, J-statistics and p-values.

SDF	K-N CAPM	K-S CAPM	CAPM+SMB	C-CAPM	EHB	LRR
Abnorm.perf. (per $1)	0.25	0.15	0.38	0.14	0.13	0.10
			SDF parameters			
a	0.161	0	0.276			
	(0.01)		(0.01)			
b	3.63	1	4.00	2.01	3.23	8.01
	(0.14)		(0.17)	(0.04)	(0.15)	(0.80)
J–stat.	2.46	1.21	0.57	4.16	5.34	9.52
$P(\chi^2_1 > J)$	0.11	0.23	0.31	0.00	0.00	0.00

beliefs into the analysis. I repeat the analysis in subsection 3.1 for all considered models.

3.2.2 Excess Cash Flows and Different Beliefs.

Here, I inspect how subjective beliefs' about excess cash flows vary for different assumptions about PE investors' risk preferences. Figure 5 presents *belief gaps* for aggregate PE cash flows across different models. Generally, a similar pattern of expectations about PE cash flows as for beliefs estimated in subsection 3.1 can be observed.

Table 5 shows that the belief gap is negative most of the time for all considered models. Beliefs extracted for EHB and LRR SDFs show the smallest error, only 2% of the time; they are more pessimistic than the actual PE cash flow (52% of underexpectations). However, the negative expectations about PE cash flow for them are much bigger than the positive ones. For the EHB model, the ratio of negative-positive gaps is 13.81$ million to 7.8$ million. For the LRR model, this ratio is 16.26$ million to 9.2$ million. Unlike the beliefs extracted for K-N SDF, pessimism is not concentrated around disastrous events.

What can explain the pessimistic perception of PE funds at the beginning of the sample? The first explanation is the investors' euphoric moods about the public market at the beginning of the sample, see Figure 6, could potentially decrease the relative attractiveness of the PE industry, which lead to a negative *belief gap*. The second explanation is the so-called 'regret aversion' or

Figure 4: Pricing Errors. The gure plots the period-wise errors of average excess cash ow pricing using di erent SDFs for 1996 Q1 { 2014 Q4. The legend of the plot presents considered SDFs and MSE (mean-squared error) for each of them. All numbers is $ millions.

Table 5: Belief Gap for PE Cash Flows. The table presents some belief gap statistics for PE cash ows visualised in Figure 5.

	Belief Gap for Beliefs estimated under:					
	K-N CAPM	K-S CAPM	CAPM+SMB	CCAPM	EHB	LRR
Underexepctations, % time	63%	55%	58%	56%	52%	52%
Max Gap, $ millions	18.32	15.44	22.98	13.8	7.8	9.2
Max Gap - YQ	2008 Q4	2004 Q2	2008 Q4	2004 Q2	1999 Q3	1999 Q3
Min Gap, $ millions	-25.21	-25.12	-26.92	-21.66	-13.41	-16.26
Min Gap - YQ	2009 Q3	2005 Q4	2009 Q3	2005 Q4	1996 Q3	1996 Q3

`loss-aversion'. Investors often tend to make regret-minimising choices, as they are more sensitive to potential loss than to potential win. For example, see Zeelenberg, Beattie, Van der Pligt, and De

Vries (1996) and Qin (2015). In this light, the PE industry is known for its non-normal distribution with fat right tails in generating fund performance. It is hard to detect a good PE fund that would find new "Googles" and deliver astronomical returns. According to Metrick and Yasuda (2011), for VC funds, 43.7% of first-round investments return nothing, 38.8% of second-round investments return nothing, and 33.7% of later-round investments return nothing, while the chances of earning a multiple of five or more decrease proportionally from 19% to 7%. The substantial heterogeneity across return-generating skills among fund managers also applies to funds with BO strategies, see Korteweg and Sorensen (2017).

The further evolution of investors' expectations shows that the successful era for PE funds during the dot-com bubble has flipped the opinions into more optimistic. The EHB and LRR investors expressed immense optimism during the period when the dot-com bubble was inflating (the late 1990s). The peak was observed in 1999 Q3, a few quarters before the bubble started bursting and the followed-up recession (2001 Q2-Q3).

Another interesting observation from Figure 5 is that the crisis of 2008 does not affect the investors' expectations so strongly as for Korteweg and Nagel investor, compare with Figure 3[6]. The disagreement in results raises an important question: what was more important for investors, the dot-com bubble or the 2008 crisis? Ning, Wang, and Yu (2015) find that the impact of the 2008 financial crisis on the venture industry was less dramatic than that of the 2000 high-tech crash, possibly because venture firms are still on the way to recovering from the dramatic downfall from the 2000 peak, and this adjustment continued to the 2008 financial crisis. The subjective beliefs extracted for EHB, LRR and C-CAPM investors validate their results for PE funds.

Concerning PE cash flows, I conclude that the pessimism pattern is sustained for beliefs extracted under various assumptions on risk preferences. The periods of high expectations are followed by periods of lower expectations when the bigger pessimism replaces the optimism. The role of disastrous events in forming PE investors' expectations is significant, but the clear pattern is not determined across all subjective beliefs.

Figure 6 presents belief gaps for aggregate CRSP cash flows across different models. As in the baseline analysis (subsection 3.1), beliefs across all models agree that the investors' expectations are systematically overoptimistic about the ability of the public market to generate good cash flows. Table 6 shows that the belief gap is similar across models in terms of min max variation. For instance, EHB and LRR investors' expectations about the CRPS index after 2005 remain systematically overoptimistic. Still, the belief gap's magnitude is significantly lower than for beliefs extracted under other SDFs. For EHB, LRR and C-CAPM investors, the beliefs were not severely

[6] Only for beliefs extracted under CAPM+SMB SDF belief gap reaches its extreme values during GFC.

Figure 5: Aggregate PE Cash Flows: Belief-Adjusted vs. Observed. The gure plots the di erence between belief-adjusted and observed PE fund's aggregate cash ows (Belief Gap) for 1996 Q1 { 2014 Q4. The beliefs are model-implied subjective beliefs for di erent risk preferences. The title of each plot indicates under which SDFs beliefs are derived.

Figure 6: Aggregate CRSP Cash Flows: Belief-Adjusted vs. Observed. The figure plots the difference between belief-adjusted and observed CRSP-mimicking funds' aggregate cash flows (Belief Gap) for 1996 Q1 – 2014 Q4. The beliefs are model-implied subjective beliefs for different risk preferences. The title of each plot indicates under which SDFs beliefs are derived.

affected by GFC, unlike Korteweg and Nagel (with and without SMB factor) and Kaplan and Schoar investors.

Table 6: Belief Gap for CRSP Cash Flows. The table presents some belief gap statistics for CRSP cash flows visualised in Figure 6.

	Belief Gap for Beliefs estimated under:					
	K-N CAPM	K-S CAPM	CAPM+SMB	CCAPM	EHB	LRR
Overexepctations, % time	64%	72%	70%	71%	69%	70%
Max Gap, $ millions	32.62	26.94	40.94	30.80	29.05	32.09
Max Gap - YQ	2008 Q4	1998 Q3	2008 Q4	1998 Q3	1998 Q3	1998 Q3
Min Gap, $ millions	-6.19	-5.72	-8.51	-7.73	-6.23	-7.96
Min Gap - YQ	2009 Q2	2005 Q3	2009 Q2	1998 Q4	1998 Q4	1998 Q4

However, the optimism spikes before the dot-com bubble across all models. As we can see from Table 6, for EHB, LRR and C-CAPM models, these spikes are associated with the biggest belief gaps. They occurred around 1998 Q3. For other models, the dot-com bubble period is also associated with significant bursts of optimism. The potential explanation for this behaviour is that investors tend to form more positive expectations in the boom market, see Kieren, Müller-Dethard, and Weber (2021). The PE investor who chooses between investing in the fast-growing publicly traded companies or PE funds was apparently influenced by unprecedented growth in the NASDAQ index, whose stocks are believed to resemble the PE funds (Driessen, Lin, and Phalippou (2012)). As NASDAQ started booming in 1995, it could cause investors to form overoptimistic subjective beliefs about the public market. This effect is not so significant for PE funds' cash flows (Figure 6), because the valuation of PE funds cannot be observed frequently. Consequently, investors did not update their beliefs drastically.

Overall, 60-70% of the time, PE investors are significantly more optimistic about public market cash flows than historically observed. While 50-60% of the time, PE investors are considerably more pessimistic about PE cash flows. I can conclude that the primary source of positive abnormal performance for PE funds unexplained by the mainstream asset pricing models is the discrepancy of investors' beliefs with those implied by the Rational Expectation hypothesis. I formally test this statement in the following subsection by conducting the counterfactual exercise using estimated beliefs.

3.2.3 Counterfactual Evidence. In this subsection, I try to understand the counterfactual cash flows implied by the alternative probability distributions. Julliard and Ghosh (2012) do a similar exercise to assess the empirical plausibility of the rare disasters hypothesis in explaining the equity premium puzzle. They use the alternative probability distribution to construct the counterfactual distribution of rare disasters (size and frequency). Here I use the alternative probability distribution to construct counterfactual average risk-adjusted excess cash ow and its components using estimated beliefs. The results of the bootstrap exercise are presented in Table 7 in three di erent panels.

Table 7: Counterfactual Excess, PE and Market Cash Flows. The table presents the results of the counterfactual exercise with 100 thousand simulations in which subjective beliefs for di erent risk preferences are used as importance weights for 1996 Q1 { 2014 Q4 cash ows. Each column represents the SDF under which subjective beliefs are estimated. Panel A presents simulation results for Excess Cash Flows. The rst row reports a historical mean of discounted excess cash ows ($\overline{CF}^{exc}_{T}$). The second row reports a bootstrap mean of discounted excess cash ows ($\overline{CF}^{exc}_{boot}$). The third row reports a 95% con dence interval for the bootstrap mean of discounted excess cash ows ($[\overline{CF}^{exc}_{boot\,2:5\%};\overline{CF}^{exc}_{boot\,97:5\%}]$). The fourth row reports the probability of observing a counterfactual mean of discounted excess cash ows at least as large as the historical one ($Pr(CF^{PE}_{boot} > \overline{CF}^{PE}_{T})$). Panel B presents simulation results for PE Cash Flows, and Panel C presents simulation results for Market Cash Flows. The reported statistics are analogous to Panel A. All numbers are represented in percentages to MAP. MAP is the mean average payout, where the aggregate payout is constructed as the sum of capital distributions across all funds at every period.

	K-N CAPM	K-S CAPM	CAPM+SMB	C-CAPM	LRR	EHB
			Panel A: Excess Cash Flows (in % of MAP)			
$\overline{CF}^{exc}_{T}$	6.39	8.36	12.27	8.11	5.53	5.54
$\overline{CF}^{exc}_{boot}$	-0.02	0	0.88	0	-0.01	-0.01
$[\overline{CF}^{exc}_{boot2:5\%};\overline{CF}^{exc}_{boot97:5\%}]$	[-6.18;6.15]	[-5.73;5.74]	[-8.49;10.25]	[-5.45;5.46]	[-4.86;4.84]	[-3.99;3.97]
$Pr(CF^{exc}_{boot} > \overline{CF}^{exc}_{T})$	2.14%	0.24%	1.09%	0.2%	1.32%	0.33%
			Panel B: PE Cash Flows (in % of MAP)			
$\overline{CF}^{PE}_{T}$	-3.29	-1.12	-12.15	-2.53	-4.68	-3.07
$\overline{CF}^{PE}_{boot}$	-4.27	-3.76	-13.99	-5.23	-6.68	-5.41
$[\overline{CF}^{PE}_{boot2:5\%};\overline{CF}^{PE}_{boot97:5\%}]$	[-10.8;2.26]	[-9.88;2.37]	[-24.13;-3.85]	[-11.28;0.82]	[-11.72;-1.64]	[-9.99;-0.82]
$Pr(CF^{PE}_{boot} > \overline{CF}^{PE}_{T})$	39.05%	19.98%	37.47%	19.11%	21.77%	15.88%
			Panel C: Market Cash Flows (in % of MAP)			
$\overline{CF}^{mrkt}_{T}$	-9.68	-9.48	-24.43	-10.65	-10.21	-8.62
$\overline{CF}^{mrkt}_{boot}$	-4.26	-3.76	-14.87	-5.23	-6.67	-5.4
$[\overline{CF}^{mrkt}_{boot\,2:5\%};\overline{CF}^{mrkt}_{boot\,97:5\%}]$	[-12.19;3.68]	[-10.95;3.43]	[-29.4;-0.35]	[-12.13;1.67]	[-12.31;-1.03]	[-10.36;-0.44]
$Pr(CF^{mrkt}_{boot} > \overline{CF}^{mrkt}_{T})$	90.43%	93.89%	89.54%	93.63%	88.84%	89.53%

Panel A presents the results of the counterfactual exercise for excess cash ows. Here I compare the historical mean of discounted excess cash ows with the bootstrap mean. The bootstrap mean for all subjective beliefs is virtually zero. The exception is CAPM SDF augmented with SMB factor. CAPM+SMB SDF is the least precise pricing kernel in terms of pricing error (Figure 4), and the subjective beliefs estimated for this SDF seem to produce the least leptokurtic bootstrap distribution of excess cash ows, see Figure 7. As a consequence, the 95 % con dence interval for excess cash ows is the broadest for this distribution (third row, Panel A, Table 7). Even though, the counterfactual mean for beliefs extracted under CAMP+SMB SDF is lesser than 1% of Mean Average Payout (MAP).

Meanwhile, the SDF valuation under the Rational Expectation (RE) hypothesis (rst row, Panel A, Table 7) produce the economically signi cant non-zero means for excess cash ows. The historical mean of risk-adjusted excess cash ows varies from 5.53% for LRR SDF to 12.27% for CAPM+SMB. The striking di erence between the historical and bootstrap average signi es a con ict between beliefs implied by the RE hypothesis and subjective beliefs observed in the market and agreed to PE funds' valuation.

The third row of Panel A tests the plausibility of using the RE hypothesis in the context of PE valuation. The 95 % con dence interval for the counterfactual excess cash ows across all models does not include the historically observed mean. It means that we should reject the RE hypothesis when valuing PE investments. Thus, I can conclude that belief discrepancy is the main source of pricing error for PE funds. The SDF misspeci cation cannot solely o er a plausible explanation for the observed outperformance of PE funds. The valuation of PE funds under subjective beliefs gives the zero risk-adjusted excess cash ow in simulations that resolve the emerging valuation puzzle.

The fourth row of Panel A shows that investors price PE funds presuming a very low probability of strong excess cash ows. On average, the probability of observing a counterfactual excess cash ow at least as large as the historical one is below 3% across all models. In the below panels, I investigate the counterfactual implications for the components of the excess cash ow.

Panel B and Panel C of Table 7 present the counterfactual distribution of components of excess cash ows. Note that both bootstrap and historical excess cash ows (their averages) are equal to the di erence between their PE and market counterparts (the rst two rows of the below panels). The third row of Panel B shows that the historical PE cash ow is in a 95% con dence interval. Similarly, the empirically realised average market cash ow is also plausible across all models. Therefore, the counterfactual experiment witnesses that, separately, both components are very plausible; however, the likelihood of their combination that leads to the historical excess cash ow is not so.

We can refer to the fourth row of each panel to investigate why the RE hypothesis is rejected

Figure 7: Bootstrap Density of Average Discounted Excess Cash Flows. The gure plots the bootstrap density of counterfactual excess cash ows implied by six types of investors indicated in the legend of this graph.

for excess cash ows. With a probability of more than 88%, investors expect a higher market cash ow than occurred in reality. Meanwhile, with a probability not higher than 40%, investors expect a higher PE cash ow. As we saw in Figures 5 and 6, investors exert more optimism toward the market cash ows rather than the PE ones. Hence, the bootstrap exercise provides counterfactual evidence in favour of the `PE pessimism { market optimism' solution for the PE valuation puzzle.

In summary, subjective beliefs help explain abnormal PE excess cash ows. I can reject the RE hypothesis for the valuation of PE funds with signi cant statistical con dence. The outperformance exists because investors tend to overestimate the performance of the public market and underestimate the performance of PE funds. As a result, investors might want to demand an excessive premium for holding PE stakes. These ndings are consistent across all mainstream asset pricing models for which I estimate the investors' beliefs.

There remain two unaddressed concerns so far. Question number one is how I can ensure that the beliefs I estimate represent the investors' sentiments. The following section tackles this concern. Question two concerns investors' (LPs) ability to translate their beliefs into a price discount. Appendix B tackles this concern.

4 Analysis of Beliefs

I have shown that we can back out subjective beliefs that help explain the PE valuation. However, whether the extracted alternative probability distribution corresponds to surveyed investor beliefs remain open. In this section, I will prove that the subjective beliefs are squared with the survey data. Such evidence is in sub-section 4.2. In the next sub-section 4.1, I explain the publicly available survey data I use to validate my results. Finally, in sub-sections 4.3,4.4, I address the concerns if model-implied subjective beliefs are sensitive to SDF misspeci cation and if used PE sentiment indices adequately re ect the investors' opinions.

4.1 Survey Data

I consider survey data for three types of investors: individual, institutional and Private Equity investors. The individual investors got signi cant attention in Greenwood and Shleifer (2014) paper. I borrow two surveys from their paper: 1) the Gallup survey (Gallup), conducted monthly between 1996-2012 with occasional gaps 2) Robert Shiller's survey of wealthy individual investors (Shiller (Ind)) available sporadically between 1999 and July 2001 and monthly afterwards. The opinions of institutional investors are represented by Robert Shiller's institutional investor survey (Shiller (Inst)) conducted at six-month intervals from July 1989 to July 2001 and monthly afterwards.

Table 8: Sentiment Indices Correlation. The table reports the correlations between Institutional (Shiller (inst)), Individual (Shiller (ind), Gallup, ESI) and PE (CEPECI, SVVCCI) sentiment indices.

	Shiller (inst)	Shiller (ind)	Gallup	ESI	CEPECI	SVVCCI
Shiller (inst)	—	0.51***	-0.52***	-0.55***	0.09	-0.05
Shiller (ind)	0.51***	—	-0.10	-0.17	0.57***	0.43***
Gallup	-0.52***	-0.10	—	0.78***	0.90***	0.83***
ESI	-0.55***	-0.17	0.78***	—	0.86***	0.76***
CEPECI	0.09	0.57***	0.90***	0.86***	—	0.82***
SVVCCI	-0.05	0.43***	0.83***	0.76***	0.82***	—
Note:						$p<0.1$; $p<0.05$; $p<0.01$

The Silicon Valley Venture Capital Con dence Index (SVVCCI) and Central Europe Private Equity Con dence Index (CEPECI) re ect the opinions of investors in PE. There does not exist more granular data for investors in PE funds concentrated on BO strategy. Figure 8 presents the time series for PE indices and shows that the aggregated sentiments do not di er too much. The moods of PE and VC investors in the San Francisco area and Central Europe have a signi cant correlation of 0.82, see Table 8.

Figure 8: Expectations of PE investors. The gure plots investors' expectations about the private equity market in Central Europe (blue line) and San Francisco Bay area (brown line). For Europe, the con dence index is represented by a survey of PE professionals conducted by Deloitte, while investors' moods of Silicon Valley VC capitalists are represented by a survey conducted by the University of San Francisco.

The SVVCCI is based on a recurring quarterly survey (since Q1 2004) of San Francisco Bay Area venture capitalists. The Index measures and reports the opinions of professional venture capitalists on their estimations of the high-growth venture entrepreneurial environment over the next 6 { 18 months. Please see Appendix 2, Cannice and Goldberg (2009) for details of the quarterly survey.

The CEPECI is a half-year survey conducted by Deloitte of PE professionals focused on Central Europe, see Deloitte website. The index is composed of answers to the rst seven questions of the survey. These questions focus on: Economic climate, Debt availability, Investors' Focus, Size of transactions, Market activity, Investment return, and Investors' activities. The average of positive answer ratios over the sum of positive and negative answers is computed for each period. This average is compared to the base period { spring 2003.

Further, I investigate how the market expectations comove across di erent types of investors. Figure 9 plots the evolution of three indices: Shiller Individual, Shiller Institutional and CEPECI. The sentiment about PE and the public market frequently dissonate. Moreover, even institutional and individual investors do not always agree about the movement of the aggregate stock market; see also a correlation in the rst column of Table 8. The literature con rms this point of view; for example, see Ghosh, Korteweg, and Xu (2020) for an analysis of subjective beliefs of the

Figure 9: Expectations: PE vs. Institutional vs. Individual. The figure plots investors' expectations about the stock market from the U.S. Institutional One-Year Confidence Index of Shiller's Institutional (pink line) and Individual (violet line) Investors (the left scale). Also, the expectations about the private equity market in Central Europe (blue line) are plotted (the right scale).

heterogeneous public market investors.

Also, it is essential to note that disagreements might happen even within the same type of investors, as in the example of Gallup and Shiller Individual. The pairwise correlation between them is low and negative (-0.1) but statistically insignificant. The potential explanation is that the investors are less homogeneous than it seems: Adam and Nagel (2022) show that Shiller index surveys wealthy individuals, contrary to Gallup who surveys the average one. Furthermore, the results might vary depending on the sample choice, even for longer time series, see Greenwood and Shleifer (2014).

4.2 Model-Implied Beliefs and Investors' Sentiment

The main question: do beliefs extracted from the model match with survey data? To answer this question requires understanding the mechanism of how the expectations might affect the excess cash flows. The most crucial point is that investors are asked in questionnaires to forecast cash flows over some distant future. To pin down this feature of sentiment indices, I consider them in Table 9 with a one-year lag. Another point is to understand subjective beliefs I estimate using my approach are about the excess market cash flow. The sentiments documented in survey data might a ect the excess cash ows not directly but via PE and stock market components.

Table 9: Correlation between Excess Cash Flow and Surveys. The table reports the correlations between sentiment indices (in rows) and 1) belief-adjusted excess cash ow (Panel A) 2) Placebo test in which the term structure of investors' expectations is not accounted (Panel B). Subjective beliefs are estimated using di erent models (in columns).

	Panel A: Average Excess Cash Flow adjusted for Beliefs estimated for:					
	K-N CAPM	K-S CAPM	SMB+CAPM	C-CAPM	EHB	LRR
Shiller (inst)	-0.25**	-0.18	-0.18	-0.12	-0.08	-0.11
Shiller (ind)	-0.02	-0.02	-0.02	0.03	0.08	0.08
Gallup	0.45***	0.36***	0.36***	0.29**	0.26**	0.29**
ESI	0.39***	0.32***	0.32***	0.26**	0.24**	0.26**
CEPECI	0.60***	0.57***	0.57***	0.57***	0.57***	0.58***
SVVCCI	0.73***	0.70***	0.70***	0.69***	0.68***	0.68***
	Panel B: Placebo Test for Contemporaneous Correlations					
Shiller (inst)	-0.16	-0.14	-0.14	-0.10	-0.05	-0.07
Shiller (ind)	-0.13	-0.17	-0.17	-0.13	-0.07	-0.08
Gallup	-0.03	-0.11	-0.11	-0.16	-0.19	-0.16
ESI	0.33***	0.23**	0.23**	0.16	0.13	0.17
CEPECI	0.14	0.06	0.06	0.05	0.07	0.10
SVVCCI	0.29*	0.21	0.21	0.19	0.21	0.24
Note:				p< 0.1;	p< 0.05;	p< 0.01

Let us focus rst on PE survey data, represented by CEPECI and SVVCCI indices. For example, San Francisco VC investors forecast the \future high-growth venture entrepreneurial environment" on a scale from one to ve. I interpret their optimism (pessimism) as the subjective probability of occurring of a good (bad) PE cash ow and consequently good (bad) excess cash ow. Therefore, I expect the survey indices positively correlate with one-year lead belief-adjusted average excess and PE cash ows. Panel A reports results for belief-adjusted excess cash- ows that validate that survey data match with subjective beliefs. The correlation is high and statistically signi cant. When comparing these results with 'non-specialised' sentiment, PE investors' opinions play the most prominent role in predicting PE excess cash ows.

Further, I inspect the individual and institutional sentiment indices. At rst glance, we see the puzzling nding: individual investors correctly predict the belief-adjusted cash ows, while the sentiment index for institutional investors misdirects it. The correlation for Shiller (inst) is negative but mostly insigni cant except Korteweg and Nagel model. It means that institutional sentiments still might a ect PE excess cash ows, but to a lesser extent than PE and individual investors.

Greenwood and Shleifer (2014) find that individual investors' expectations in survey data are inconsistent with the rational expectations model. They show that individual investors' expectations usually translate into average negative stock returns. Note that the mechanism is different from PE investors. The optimism of individual investors translates into lower public market cash flow (misprediction) and consequently into higher excess cash flow. That is why the positive correlation is observed. On the contrary, Ghosh, Korteweg, and Xu (2020) find that institutional investors, being countercyclical in expectations, correctly predict the aggregate equity market. Thus, their positive expectations translate into positive public market cash flows, diminishing the excess cash flows. It explains the negative correlation sign.

Table 9, Panel B reports the results of the placebo test. Here I conduct the correlation analysis for contemporaneous levels of sentiments and belief-adjusted excess cash flows. In Panel A, I choose uniformly for all indices the lag of one year as the standardisation technique. However, for example, San Francisco VC index reports the opinions over the next 6 – 18 months, and the European PE confidence index focuses on expectations for the `upcoming months`. In the placebo test, I do not account for the potential lingering between the time of answers and the time when it affects cash flows. Panel B reveals no correlation for extracted beliefs except four pairs: CEPECI – K-N CAPM, ESI – K-N CAPM, ESI – K-S CAPM, and ESI – SMB+CAPM. Recall from sub-section 3.2 that the subjective beliefs back out under these models were the least accurate. Therefore, less noise induced by SDF misspecification leads to a more precise estimation of subjective beliefs.

In summary, I show that subjective beliefs correspond to survey data. The opinions of PE investors are the most important for predicting PE excess cash flows. The beliefs of individual and institutional investors affect excess cash flows via the public market component. The placebo test indicates potential concerns with regard to the SDF misspecification and its influence on estimated beliefs. The following sub-section addresses this concern.

4.3 Beliefs Adjusted for SDF Misspecification

If risk preferences are not adequately reflected in SDF, it can lead to high pricing errors. The noise associated with model errors might distort beliefs. In this subsection, I test the potential sources of risks persistent in the public equity market and not captured by SDFs used in my analysis.

4.3.1 Public Market Conditions. Mainly I investigate the market fundamentals as a potential source of omitted factors. In Table 10 I test expected market premia (*expected premia*) from Haddad, Loualiche, and Plosser (2017), Moody's BBB to AAA credit spread (*credit spread*), Chicago Board Options Exchange S&P 100 Volatility Index (*vxo*), aggregate liquidity factor of Pastor and Stambaugh (2003) (*liquidity*). This exercise aims to 'purify' the model-implied beliefs estimated in the previous section. I will use the residuals from all six regressions to construct the 'purified' beliefs, which means beliefs adjusted for the potential SDF misspecification. Additionally, Table 10 can reveal the information about the most plausible public market factors missed in SDFs considered in this article.

Table 10: Subjective Beliefs and Market Fundamentals. The table reports the regression analysis of model-implied subjective beliefs against market fundamentals. The market fundamentals are presented by expected market premia, credit spread, volatility index, aggregate liquidity factor and their contemporaneous changes (Δ_t).

	Model-Implied Beliefs for:					
	K-N CAPM	K-S CAPM	CAPM+SMB	C-CAPM	EHB	LRR
	(1)	(2)	(3)	(4)	(5)	(6)
expected premia	0.0001	0.0003	0.0003	0.0003	0.0001	0.0001
	(0.0001)	(0.0002)	(0.0002)	(0.0002)	(0.0002)	(0.0001)
Δ_t(credit spread)	0.008	0.001	0.001	0.004	0.007	0.006
	(0.003)	(0.005)	(0.005)	(0.005)	(0.005)	(0.004)
Δ_t(vxo)	0.011	0.015	0.015	0.014	0.011	0.010
	(0.003)	(0.004)	(0.004)	(0.005)	(0.005)	(0.004)
Δ_t(liq.agg)	0.004	0.012	0.012	0.012	0.009	0.010
	(0.006)	(0.009)	(0.009)	(0.010)	(0.010)	(0.008)
Constant	0.013	0.011	0.011	0.012	0.013	0.013
	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)
Observations	75	75	75	75	75	75
R^2	0.412	0.313	0.313	0.219	0.132	0.159
Adjusted R^2	0.378	0.273	0.273	0.174	0.083	0.111
F Statistic	12.248	7.956	7.956	4.909	2.667	3.313
Note:				$p<0.1$;	$p<0.05$;	$p<0.01$

For all subjective beliefs the changes in the market volatility show statistically significant effect. Gompers, Kovner, Lerner, and Scharfstein (2008) show that VC firms increase their investments during periods of high volatility in the public market with no real degradation in their performance.

The Limited Partners react in a similar fashion by piling more capital into Private Equity[7]. Increasing volatility in the public market signifies that public market cash flow is contracting. All else equal, it leads to an increase in excess cash flow. Therefore, the changes in volatility expectedly positively correlate with average excess cash flow and negatively with subjective beliefs.

The next important factor is the credit spread. There is evidence in the literature that both factors might influence investors' decisions. For example, Schmid, Huether, and Steri (2019) investigate the opportunity to price private firms by using the information about their loan portfolio. They come up with the so-called credit market equivalent metric and show the importance of the credit market returns on PE valuation. Increasing spread signifies the lower public market cash flow and consequently higher excess cash flow. Only beliefs estimated for Korteweg and Nagel investor demonstrate the expected (negative) sign and the statistical significance for this variable.

The expected market premia can play an important role in PE excess cash flows. Haddad, Loualiche, and Plosser (2017)investigate the reaction of the PE industry to the changes in the aggregate risk premium and find that a low risk premium increases the present value of performance gains and decreases the cost of holding an illiquid investment. Increasing expected market premia signifies the higher public market cash flow and consequently lower excess cash flow. Kaplan and Schoar and Korteweg and Nagel plus SMB models have positive and significant coefficients for this variable. On the whole, the credit market conditions and expected market premia can be among the omitted risk factors, but their importance is lower in comparison with the market volatility factor.

Another interesting fact is the aggregate market liquidity (Pastor and Stambaugh (2003)) seems to have no impact on subjective beliefs. The extensive literature supports the evidence favouring liquidity as highly important in PE, for example, Franzoni, Nowak, and Phalippou (2012). The regression table shows the expected sign for all models, though they are insignificant. The effect of aggregate market liquidity might be eclipsed by the other public market conditions considered here.

Can we come up with a new SDF that does a better valuation job? Probably, yes. Overall, I conclude that potentially changes in market volatility together with expected market premia and credit conditions might be missed in the existing asset pricing models applied to PE cash flows. How critically the potential SDF misspecification affects the estimated subjective beliefs? Can we still trust results? Let us use the residuals from presented regressions as 'purified' (orthogonalized) beliefs to construct belief-adjusted cash flows and repeat validation as in subsection 4.2.

[7] June 1, 2022, Altexchange.com; May 26, 2022,Wall Street Journal

4.3.2 'Puri ed' Beliefs. Further, I want to check if the observed correlation in Table 9 persists when using 'puri ed' beliefs. From Table 10 I back out the residuals, and they will be `puri ed` beliefs, in the sense that I orthogonalize them to market conditions which are potentially omitted in presented risk preferences. Then I rescale the residuals' time series to satisfy the beliefs conditions: sum to one and non-negativity. In Figure 10, 'puri ed' beliefs strongly correlate with the model-implied ones, exhibiting that the beliefs extracted using my methodology capture rather investors' opinions than misspeci cation noise. For beliefs estimated under other pricing kernels the correlation is the following: K-S CAPM { 0.83; CAPM+SMB { 0.83; CCAPM { 0.89; LRR { 0.92; EHB { 0.93. A higher correlation is observed for subjective beliefs estimated under the SDFs, which produce a lower pricing error for excess cash ows. It signi es that more calibrated SDF outputs more informative beliefs.

Figure 10: Subjective Beliefs: Raw vs. Puri ed. The gure plots the model-implied subjective and orthogonalized beliefs obtained after the projection of raw beliefs to public market conditions (Table 10) for Korteweg-Nagel SDF for 1996 Q1 { 2014 Q4.

Then I repeat the exercise from Table 9, but now, I use beliefs adjusted for SDF misspeci cation ('puri ed') instead of 'raw' ones. The results remain similar: the strongest correlation of beliefs is with PE survey indices. The statistical signi cance of the correlation is not altered at all, while the magnitude gets intangibly lower. The correlation analysis for individual and institutional investors has not changed signi cantly too. Therefore, I conclude that my methodology robustly estimates the investor beliefs from observed pricing errors. SDFs misspeci cation does not critically distort beliefs and their implications.

Table 11: `Puri ed` Beliefs and Survey Data. The table reports the correlations between sentiment indices (in rows) and belief-adjusted excess cash ow. Subjective beliefs are estimated using di erent models (in columns) and then orthogonalized to public market conditions (Table 10).

	Average Excess Cash Flow adjusted for Puri ed Beliefs estimated for:					
	K-N CAPM	K-S CAPM	SMB+CAPM	C-CAPM	EHB	LRR
Shiller (inst)	-0.24**	-0.18	-0.18	-0.11	-0.08	-0.10
Shiller (ind)	-0.01	-0.01	-0.01	0.03	0.08	0.08
Gallup	0.42***	0.36***	0.36***	0.30**	0.28**	0.30**
ESI	0.38***	0.33***	0.33***	0.28**	0.26**	0.28**
CEPECI	0.60***	0.58***	0.58***	0.58***	0.58***	0.58***
SVVCCI	0.71***	0.69***	0.69***	0.67***	0.66***	0.67***

4.4 PE Market Conditions and Investors' Sentiment

Additionally to the problems associated with recovered beliefs, I investigate if the sentiment indices on their own make economic sense. Mainly, I want to test if PE market conditions correlate expectedly with investors' opinions and if it nds support in the literature. I consider the entry-exit conditions and secondary market transactions in the PE market. Table 12 presents the results of correlation analysis for these factors with one-lag ahead subjective beliefs.

The correlations for all indices demonstrate that the entry conditions presented by fundraising activity signi cantly a ect the investors' expectations. It is important to note that high fundraising translates into pessimistic expectations of PE investors. Brown et al. (2021) nd that periods of increased fundraising have been followed by periods of low performance. This mechanism leads that fundraising activity negatively a ects the investors' opinions about future cash ows. The correlation for absolute fundraising level calculated as a quarterly sum of capital calls for PE funds validates this evidence.

An active secondary market also negatively a ects the opinions of PE investors. After active deal-making on the secondary market, investors expect weak cash ows. The explanation might be the following: if investors see an increasing number of investors wishing to leave the market, they become less con dent about the PE funds' ability to generate cash ows in the future. This hypothesis nds support in the literature; for instance, Nadauld, Sensoy, Vorkink, and Weisbach (2019) show that buyers in the secondary market outperform sellers by supplying liquidity to investors wishing to leave. Therefore, the number of transactions on the secondary market can

Table 12: PE Market Conditions and Sentiment. The table reports the correlations between lagged conditions on PE market (rows) and model-implied subjective beliefs (columns). The PE conditions are presented by the amount of raising money by PE funds from Preqin sample (fundraise), the number of secondary transactions (scnd. transactions), the number of BO IPO deals (#BO IPO deals) and the average size of exit deals (avg. BO exit size) and their contemporaneous changes (Δ_t). IPO data is provided by Ritter's website.

	CEPECI	SVVCCI
fundraise	-0.55***	-0.34**
Δ_t(fundraise)	0.17	0.14
scnd. transactions	-0.35**	-0.14
Δ_t(scnd. transactions)	0.03	0.03
#BO IPO deals	0.17	0.37**
Δ_t(#BO IPO deals)	0.24	0.23
avg. BO exit size	-0.19	0.04
Δ_t(avg. BO exit size)	0.02	0.11

indicate the bad state of the PE market and consequently lead to a negative correlation with investors' sentiment. The results presented in Table 12 confirm this view.

Conventionally the IPO market activity serves as a barometer for investors' sentiment. Good exit conditions make investors form positive expectations. Nanda, Samila, and Sorenson (2020) find that each additional initial public offering (IPO) among a VC firm's first ten investments predicts as much as an 8% higher IPO rate on its subsequent investments. Table 12 shows that VC investors' moods represented by the SVVCCI index positively correlate with an active IPO market. Note that the number of IPOs is more influential than the size of exit transactions.

In summary, I show that PE survey data serving to validate the extracted subjective beliefs logically correlate with PE market conditions. Active fundraising and the market of secondary transactions negatively affect the expectations of PE investors, while a good IPO market makes them more enthusiastic about the industry. These findings find support in the literature.

5 Conclusion

The valuation of non-frequently traded assets remains an open challenge. This paper shows that the standard SDF approach is insufficient to explain PE valuation. The fact that the prices reflect not only risk preferences but also investor beliefs is overlooked in the existing performance metrics. In this paper, I propose a general methodology to estimate investors' unconditional beliefs. I demonstrate that investors' subjective beliefs can explain valuation of PE funds. Estimated beliefs show that the observed positive abnormal performance of funds is associated with the fact that investors systematically overestimate the performance of the stock market portfolio while underestimating the PE funds' performance. I validate subjective beliefs by using survey data. Additionally, I show that SDF misspecification can not critically distort the estimated beliefs.

This methodology can be applied to other asset classes with similar characteristics, such as VC or Real Estate funds. Furthermore, incorporating the observed investors' sentiment into the asset pricing model holds potential for improving the accuracy of cash flow forecasts.

References

Adam, K., & Nagel, S. (2022). Expectations data in asset pricing (tech. rep.). National Bureau of Economic Research.

Almeida, C., Ardison, K., & Garcia, R. (2020). Nonparametric assessment of hedge fund performance. Journal of Econometrics, 214(2), 349{378.

Almeida, C., & Garcia, R. (2012). Assessing misspeci ed asset pricing models with empirical likelihood estimators. Journal of Econometrics, 170(2), 519{537.

Almeida, C., & Garcia, R. (2017). Economic implications of nonlinear pricing kernels. Management Science, 63(10), 3361{3380.

Anatolyev, S., & Gospodinov, N. (2011). Methods for estimation and inference in modern econometrics. CRC Press.

Ang, A., Chen, B., Goetzmann, W. N., & Phalippou, L. (2018). Estimating private equity returns from limited partner cash ows. The Journal of Finance, 73(4), 1751{1783.

Bansal, R., & Yaron, A. (2004). Risks for the long run: A potential resolution of asset pricing puzzles. The journal of Finance, 59(4), 1481{1509.

Barberis, N., & Thaler, R. (2003). A survey of behavioral nance. Handbook of the Economics of Finance, 1, 1053{1128.

Brown, G., Harris, R., Hu, W., Jenkinson, T., Kaplan, S. N., & Robinson, D. T. (2021). Can investors time their exposure to private equity? Journal of Financial Economics, 139(2), 561{577.

Brunnermeier, M., Farhi, E., Koijen, R. S., Krishnamurthy, A., Ludvigson, S. C., Lustig, H., Nagel, S., & Piazzesi, M. (2021). Perspectives on the future of asset pricing. The review of nancial studies.

Campbell, J. Y., & Cochrane, J. H. (1999). By force of habit: A consumption-based explanation of aggregate stock market behavior. Journal of political Economy, 107(2), 205{251.

Cannice, M. V., & Goldberg, C. S. (2009). Venture capitalists' con dence, asymmetric information, and liquidity events. Journal of Small Business & Entrepreneurship, 22(2), 141{164.

Cejnek, G., Franz, R., & Stoughton, N. (2017). An integrated model of university endowments. Available at SSRN 2348048

Chen, X., Hansen, L. P., & Hansen, P. G. (2020). Robust identi cation of investor beliefs. Proceedings of the National Academy of Sciences, 117(52), 33130{33140.

Chung, J.-W., Sensoy, B. A., Stern, L., & Weisbach, M. S. (2012). Pay for performance from future fund ows: The case of private equity. The Review of Financial Studies, 25(11), 3259{3304.

Colacito, R., & Croce, M. M. (2011). Risks for the long run and the real exchange rate. Journal of Political economy, 119(1), 153{181.

Csiszar, I. (1975). I-divergence geometry of probability distributions and minimization problems. The annals of probability, 146{158.

Driessen, J., Lin, T.-C., & Phalippou, L. (2012). A new method to estimate risk and return of nontraded assets from cash ows: The case of private equity funds. Journal of Financial and Quantitative Analysis, 47(3), 511{535.

Efron, B. (1981). Nonparametric standard errors and con dence intervals. canadian Journal of Statistics, 9(2), 139{158.

Elan, S. L., & Goodrich, M. (2010). Behavioral patterns and pitfalls of us investors. Federal Research Division, Library of Congress, 01.

Fama, E. F., & French, K. R. (1992). The cross-section of expected stock returns. the Journal of Finance, 47(2), 427{465.

Franzoni, F., Nowak, E., & Phalippou, L. (2012). Private equity performance and liquidity risk. Journal of Finance, 67(6), 2341{2373.

Ghosh, A., Julliard, C., & Taylor, A. P. (2017). What is the consumption-capm missing? an information-theoretic framework for the analysis of asset pricing models. The review of nancial studies, 30(2), 442{504.

Ghosh, A., Korteweg, A. G., & Xu, Q. (2020). Recovering heterogeneous beliefs and preferences from asset prices. Available at SSRN 3650199.

Ghosh, A., & Roussellet, G. (2020). Identifying beliefs from asset prices. Proceedings of Paris December 2019 Finance Meeting EUROFIDAI-ESSEC.

Glaser, M., & Weber, M. (2007). Overcon dence and trading volume. The Geneva Risk and Insurance Review, 32(1), 1{36.

Gompers, P., Kovner, A., Lerner, J., & Scharfstein, D. (2008). Venture capital investment cycles: The impact of public markets. Journal of nancial economics, 87(1), 1{23.

Gredil, O., Sorensen, M., & Waller, W. (2019). Evaluating private equity performance using stochastic discount factors. Available at SSRN 3506847.

Greenwood, R., & Shleifer, A. (2014). Expectations of returns and expected returns. The Review of Financial Studies, 27(3), 714{746.

Gupta, A., & Van Nieuwerburgh, S. (2021). Valuing private equity investments strip by strip. The Journal of Finance, 76(6), 3255{3307.

Haddad, V., Loualiche, E., & Plosser, M. (2017). Buyout activity: The impact of aggregate discount rates. The Journal of Finance, 72(1), 371{414.

Harris, R. S., Jenkinson, T., & Kaplan, S. N. (2014). Private equity performance: What do we know? The Journal of Finance, 69(5), 1851{1882.

Imbens, G. W., Spady, R. H., & Johnson, P. (1998). Information theoretic approaches to inference in moment condition models. Econometrica, 66(2), 333.

Julliard, C., & Ghosh, A. (2012). Can rare events explain the equity premium puzzle? The Review of Financial Studies, 25(10), 3037{3076.

Kaplan, S. N., & Schoar, A. (2005). Private equity performance: Returns, persistence, and capital ows. The journal of nance, 60(4), 1791{1823.

Keating, D. (2006). Gp con dential. Private Equity International, June, 70{73.

Kieren, P., Müller-Dethard, J., & Weber, M. (2021). Why so negative? belief formation and risk taking in boom and bust markets. Belief Formation and Risk Taking in Boom and Bust Markets (May 6, 2021).

Kitamura, Y., & Stutzer, M. (1997). An information-theoretic alternative to generalized method of moments estimation. Econometrica: Journal of the Econometric Society, 861{874.

Knight, F. H. (1921). Risk, uncertainty and pro t. book

Korteweg, A., & Nagel, S. (2016). Risk-adjusting the returns to venture capital. The Journal of Finance, 71(3), 1437{1470.

Korteweg, A., & Sorensen, M. (2017). Skill and luck in private equity performance. Journal of Financial Economics, 124(3), 535{562.

Kroencke, T. A. (2017). Asset pricing without garbage. The Journal of Finance, 72(1), 47{98.

Lerner, J., & Leamon, A. (2011). Yale university investments office: February 2011. *Harvard Business School Entrepreneurial Management Case*, (812-062).

LeRoy, S. F. (1973). Risk aversion and the martingale property of stock prices. *International Economic Review*, 436–446.

Manski, C. F. (2004). Measuring expectations. *Econometrica*, *72* (5), 1329–1376.

Metrick, A., & Yasuda, A. (2011). *Venture capital and the nance of innovation*. John Wiley & Sons.

Miller, E. M. (1977). Risk, uncertainty, and divergence of opinion. *The Journal of nance*, *32* (4), 1151–1168.

Muth, J. F. (1961). Rational expectations and the theory of price movements. *Econometrica: Journal of the Econometric Society*, 315–335.

Nadauld, T. D., Sensoy, B. A., Vorkink, K., & Weisbach, M. S. (2019). The liquidity cost of private equity investments: Evidence from secondary market transactions. *Journal of Financial Economics*, *132* (3), 158–181.

Nanda, R., Samila, S., & Sorenson, O. (2020). The persistent effect of initial success: Evidence from venture capital. *Journal of Financial Economics*, *137* (1), 231–248.

Ning, Y., Wang, W., & Yu, B. (2015). The driving forces of venture capital investments. *Small Business Economics*, *44* (2), 315–344.

Owen, A. B. (1988). Empirical likelihood ratio confidence intervals for a single functional. *Biometrika*, *75* (2), 237–249.

Pastor, L., & Stambaugh, R. F. (2003). Liquidity risk and expected stock returns lubos pastor. *111* (3).

Phalippou, L., & Gottschalg, O. (2009). The performance of private equity funds. *Review of Financial Studies*, *22* (4), 1747–1776.

Post, T., Karabati, S., & Arvanitis, S. (2018). Portfolio optimization based on stochastic dominance and empirical likelihood. *Journal of Econometrics*, *206* (1), 167–186.

Qin, J. (2015). A model of regret, investor behavior, and market turbulence. *Journal of Economic Theory*, *160*, 150–174.

Robinson, D. T., & Sensoy, B. A. (2016). Cyclicality, performance measurement, and cash flow liquidity in private equity. *Journal of Financial Economics*, *122* (3), 521–543.

Ross, S. (2015). The recovery theorem. *The Journal of Finance*, *70* (2), 615–648.

Rubinstein, M. (1976). The valuation of uncertain income streams and the pricing of options. *The Bell Journal of Economics*, 407–425.

Schmid, L., Huether, N., & Steri, R. (2019). Credit market equivalents and the valuation of private firms. *Available at SSRN 3463333*.

Sensoy, B. A., Wang, Y., & Weisbach, M. S. (2014). Limited partner performance and the maturing of the private equity industry. *Journal of Financial Economics*, *112* (3), 320–343.

Smith, R. J. (1997). Alternative semi-parametric likelihood approaches to generalised method of moments estimation. *The Economic Journal*, *107* (441), 503–519.

Stutzer, M. (1996). A simple nonparametric approach to derivative security valuation. *The Journal of Finance*, *51* (5), 1633–1652.

Zeelenberg, M., Beattie, J., Van der Pligt, J., & De Vries, N. K. (1996). Consequences of regret aversion: Effects of expected feedback on risky decision making. *Organizational behavior and human decision processes*, *65* (2), 148–158.

Appendices

A Constructing Stationary Excess Cash Flows

The ET estimator requires the underlying time series of excess cash flows to be stationary to derive the unconditional subjective beliefs. Figure A.1 visualises the problem with aggregate cash flows. The first issue is with the beginning of the sample: too few funds exist to generate cash flows. Moreover, these sparse cash flows consist mostly of capital calls, which by construction would one-to-one coincide with the cash flows of the benchmark market fund. That means no variability in excess cash flows.

A similar problem arises at the end of the sample. The funds incepted after 2015 (the last inception year in my sample) have cash flows heavily represented by valuations. It causes a spike in the excess cash flows, leading to undesirable time series statistical properties. The situation is exhibited in the top panel of Figure A.1.

The solution is to cut the time series and consider the shorter one. The middle panel of Figure A.1 shows that the time series for trimmed aggregate excess cash flows show better stationarity properties. However, prior-2000 cash flows are less volatile than post-2000 ones. The number of funds operating in the industry has been steadily growing. Therefore the aggregate excess cash flows increased between vintage of funds, leading to an increasing magnitude of excess cash flows and their volatility.

To tackle this problem, I consider the average excess cash flow in my main analysis. The bottom panel of Figure A.1 shows that the average excess cash flows look more stationary. The stationarity means that the time series does not have a trend, constant variance, autocorrelation, and seasonal pattern. Figure A.2 presents the estimation of the autocorrelation function (ACF). ACF declines to zero after one lag for average excess cash flows from 1996 Q1 to 2014 Q4. In contrast, for aggregate cash flows, the ACF drops more slowly. Thus, I show that average excess cash flows for 1996 Q1 to 2014 Q4 better fit the EL-type methodology to extract investors' beliefs.

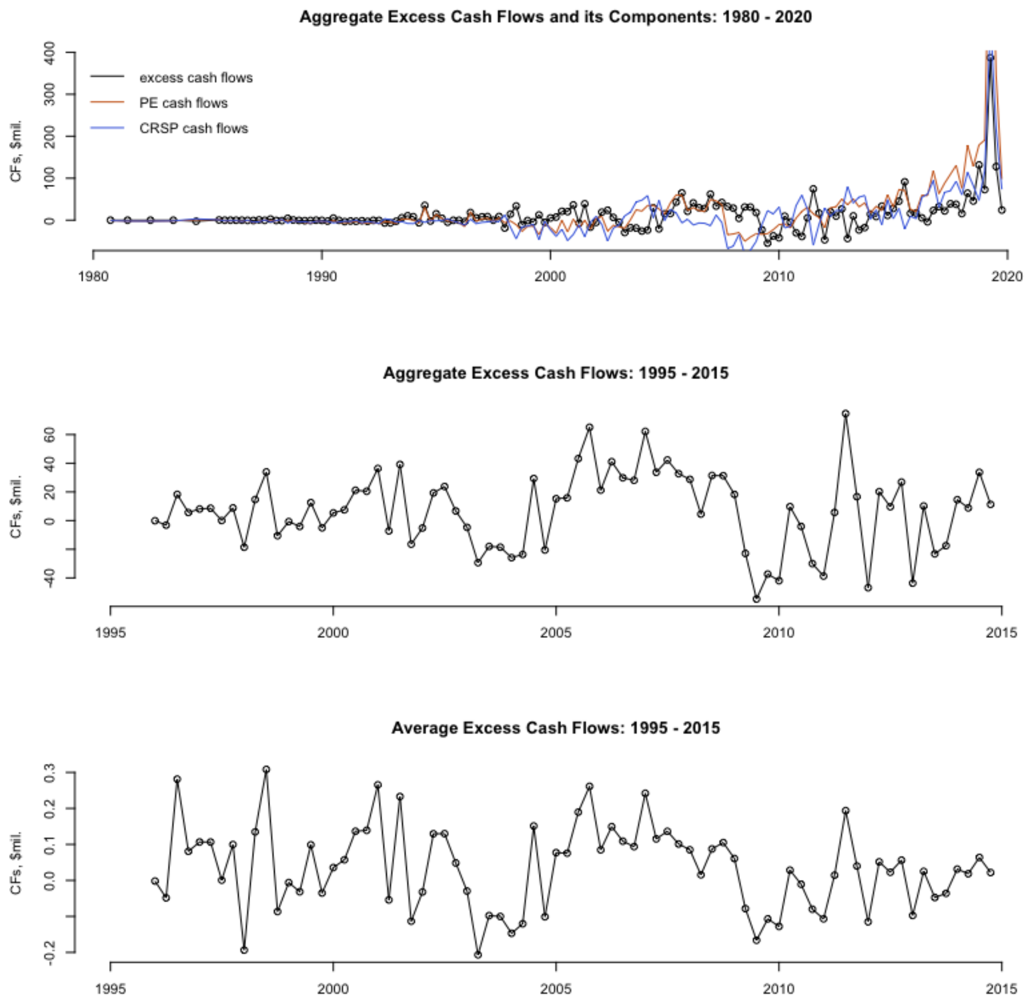

Figure A.1: Excess Cash Flows and Their Components. The figure plots the aggregate and average excess cash flow for PE funds (black line). The top panel draws the aggregate excess cash flow as the difference between aggregate PE cash flow (brownish line) and aggregate CRSP cash flows (blue line) for 1980 Q4 – 2019 Q4. The middle panel draws the aggregate excess cash flows for PE funds for 1996 Q1 – 2014 Q4. The bottom panel draws the average excess cash flows for PE funds for the same period.

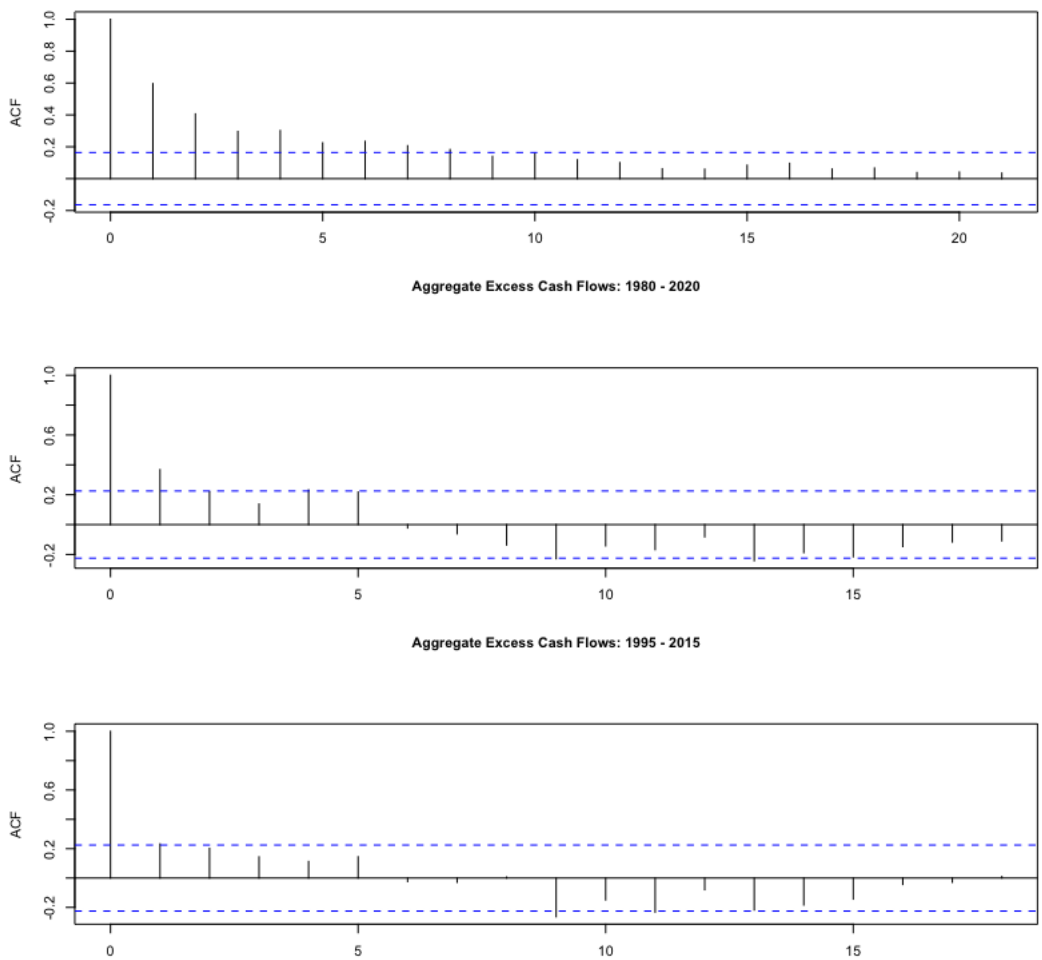

Figure A.2: Autocorrelation Function for Excess Cash Flows. The figure plots the autocorrelation function (ACF) for aggregate and average excess cash flows for PE funds. The x-axis at each plot shows the number of lags ACF is computed for. Dashed blue lines represent a zero bound.

B 'Billion Dollar Club'

This appendix investigates the capacity of Limited Partners (LPs) to translate their beliefs into price discounts within the private equity asset class. The main analysis of this paper has demonstrated the significant economic impact of beliefs on average, yet it remains uncertain whether the average investor has the bargaining power to effectively negotiate better contract conditions. In addressing this question, I restrict the sample of funds to the investments made by members of the "Billion Dollar Club" - those private equity investors who allocate $1bn or more to the asset class.

According to Preqin, there were 359 such investors in 2018, allocating a total of $1.54tn to private equity. The majority of these investors (54%) are based in North America, while 28% are based in Europe, and 9% are based in Asia and the rest of the world. It is noted by Christopher Elvin that the largest group of investors wield significant influence over the private equity industry, due to their ability to shape standards, negotiate fees, and gain access to oversubscribed vehicles.[8] This translates into their ability to secure price discounts through negotiations around fee structures or access to the best deals offered by the fund. However, it should also be noted that this concentration of bargaining power may result in disadvantages for other market participants, potentially leading to underperformance in investments made into the same funds.

Academic literature supports the notion that a General Partner's (GP) ability to raise subsequent funds and the size of those funds are dependent on the performance of their original fund (as demonstrated in studies by Sensoy, Wang, and Weisbach (2014), Kaplan and Schoar (2005), and Chung, Sensoy, Stern, and Weisbach (2012)). Kaplan and Schoar (2005) show that top-performing funds do not grow as rapidly as demand for their stakes, leading to over-subscription in such funds. In this situation, large investors such as those in the "Billion Dollar Club" have a distinct advantage in investing in champion GPs, as demonstrated by the study of Keating (2006) and the strategy implemented by David Swensen, head of the Yale University endowment, as documented in Lerner and Leamon (2011).

To construct the sample of the "Billion Dollar Club," this paper utilises Pitchbook data on LP allocation to private equity, including access to the funds LPs invest in, the size of those investments, and the date of the investment. The first member of the club was CalPERS (California Public Employees' Retirement System) in 1995, with the club growing to include 58 members by 2014.

Table ?? presents characteristics of LPs, both as members and non-members of the club, for

[8] https://docs.preqin.com/press/PE-1bn-Club-Jun-18.pdf

the years 2005 (mid-sample) and 2014 (end of sample). It can be seen that Public Pension Plans are the largest investors in buyout funds, accounting for over 50% of members of the club, and that the total commitment of the club has grown over time, now tripling their investments to reach a total of $283bn. For comparison, non-members (470 LPs) invested only $58bn in total.

On average, club members invest in larger funds, though the difference is not substantial. In the Preqin sample, the difference is more pronounced, with the average fund size for club members being almost double that of non-members. However, it should be noted that the performance of funds invested by club members is not superior. Over-subscription and privileged access for club members may create performance differences, but I find that member funds bear less left tail risk and their median fund slightly outperforms the average fund ignored by club members. It is also worth noting that non-member funds offer greater opportunities for high returns, but at a higher risk of losses. The effective time between the first capital call and the last capital distribution is shorter for club member funds (approximately 10 years) than for non-member funds (approximately 11 years). Overall, while there may be slight differences in terms of risk and performance, there is no striking difference between member and non-member funds.

Type of LP	# funds		total commitment		average fund size	
	club	non-club	club	non-club	club	non-club
2005						
Fund of Funds	2	52	11820.5	3497.2	636.4	539.8
Government Agency	1	3	4068.2	466.0	513.8	522.9
Insurance Company	2	71	2307.6	4941.8	498.3	475.0
Public Pension Fund	21	61	58302.2	9654.3	526.9	537.9
Wealth Management Firm	1	5	1059.5	451.0	440.6	608.8
Total	27	570	77558.0	39404.2	539.7	521.1
2014						
Corporate Pension	4	79	6894.0	9837.9	564.3	538.2
Endowment	3	10	7762.0	974.8	499.0	551.5
Foundation	1	41	1024.0	3126.6	627.6	504.7
Fund of Funds	4	31	12570.7	6356.7	669.4	544.5
Government Agency	1	1	7947.8	13.4	530.9	230.5
Insurance Company	4	69	10982.2	12585.6	543.2	500.5
Public Pension Fund	39	80	233139.9	18734.2	549.1	534.7
Sovereign Wealth Fund	1	1	1550.5	301.3	623.2	710.7
Wealth Management Firm	1	4	1066.6	210.0	430.1	642.1
Total	58	470	282937.7	58282.9	567.3	541.7

Beliefs. Figure **??** reveals the belief-adjusted average excess cash flows for the sub-sample of funds invested by the "Billion Dollar Club". Despite the fact that the distribution is closer to the one implied by RE hypothesis, the graph still shows a thicker negative tail and a more left-skewed distribution relative to the historical cash flows. However, the discrepancy is lower than in the full sample of funds. It's important to note that the pricing error is not eliminated. The economic impact of these errors is analyzed in subsequent figures.

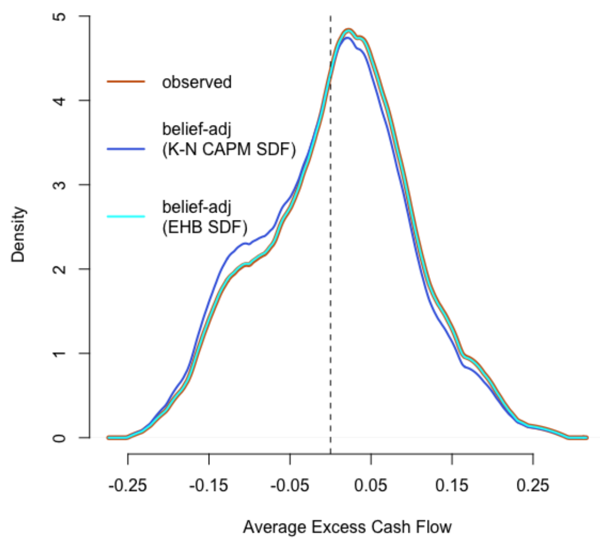

Figure **??** presents the belief-adjusted components of excess cash flow for the sub-sample of funds invested by "Billion Dollar Club" LPs. The graph reveals the belief gap for the investors with risk preferences defined by the CAPM. The "PE pessimism - Public market optimism" pattern is present, but the magnitude of the incorrect expectations about the dynamics of future cash flows is reduced compared to the full sample of funds.

Figure **??** presents the belief gap for investors with risk preferences defined by the EHB model. The 'PE pessimism - Public market optimism' pattern is evident, but the magnitude of the errors has decreased significantly. This suggests that investors in PE may have a relatively good ability to predict excess cash flows. However, even a mild degree of pessimism about PE can still result in better pricing conditions for the investor.

Table **??** highlights the positive correlation between the extracted beliefs and survey data. The results indicate that the opinions of private equity investors play a crucial role in predicting the average excess cash flow for private equity funds.

Overall, the results of the main analysis persist even when examining the sub-sample of funds invested by the "Billion Dollar Club". The findings suggest that investors systematically overes-