Towards a Normative Framework for Decumulation Investing

Lionel Martellini, Vincent Milhau and Shahyar Safaee[1]

Abstract

Individuals who have just retired with a "retirement nest egg" must decide how to spend the accumulated savings and how to invest the unspent fraction of their capital. The model of Merton (1971; Optimal Portfolio and Consumption Rules in a Continuous-Time Model; *Journal of Economic Theory*, *3*(4), 373-413) provides a utility-maximizing consumption and investment policy, which is difficult to implement in practice because it requires the specification of a risk aversion parameter and the estimation of the maximum Sharpe ratio. The "4%" spending rule, which recommends a retiree spend 4% of her initial savings plus inflation every year, is a simple and popular alternative, but it tends to underspend savings and has no guarantee of feasibility in the future. We propose a way out of this dilemma by introducing the "retirement bond", defined as an asset that pays fixed cash flows for a predefined period, and a spending rule based on the retirement bond price, which is always feasible, leaves no surplus and allows retirees to take advantage of good portfolio returns to increase withdrawals. We show that the optimal investment policy under that spending rule involves a long position in the retirement bond, and that the substitution of the standard bond with the retirement bond in traditional balanced funds and target date funds has a risk-reducing effect on replacement income.

[1] Lionel Martellini is a professor of finance at EDHEC Business School and the former director of EDHEC-Risk Institute. Vincent Milhau is a professor of finance at EDHEC Business School and a former research director at EDHEC-Risk Institute. Shahyar Safaee is the Business Development Director of Scientific Portfolio. The author for correspondence is Vincent Milhau. He can be reached at vincent.milhau@edhec.edu. The authors acknowledge the financial support of Bank of America in the context of the chair "Goals-Based Wealth Management and Applications to Retirement Investing" at EDHEC Business School. We thank Anil Suri for very useful feedback. Any remaining error is ours.

Introduction

The life cycle of an adult involves a working period followed by a retirement period, in which benefits from various pension arrangements (Social Security, defined-benefit or defined-contribution pension plans and voluntary savings) replace the labor income. Part of that replacement income is provided by systems in which the individual earned rights during her working years. Pay-as-you-go social insurance systems and defined-benefit pension funds fit in this category, but with the general trend from defined-benefit to defined-contribution, an increasing fraction of replacement income is tied to investment decisions before and after retirement and to withdrawal decisions in retirement. Individuals worried about the long-term sustainability of Social Security systems and the insufficient benefits that they will receive are also likely to engage into voluntary savings. The process of building up a retirement nest egg from periodic contributions deducted from wages is the *accumulation* stage of the life cycle, and the reverse process, which involves progressive consumption of accumulated savings to generate income in retirement is *decumulation*. Individuals in decumulation must make withdrawal decisions – how much to withdraw every year or month from the retirement pot – and investment decisions – how to invest the unspent part of their savings.

The objective of this paper is to introduce withdrawal and investment rules that are both justified by asset pricing principles and actionable in the sense that they do not require the specification or the estimation of parameters like the degree of risk aversion, the risk premia on risky assets or the parameters of stochastic models governing investment opportunities. The "actionability" criterion is important for practical implementation purposes because non-sophisticated individuals are not expected to implement proxies for optimal strategies like the one of Merton (1969, 1971), but it is equally important to propose an alternative approach that does not expose them to shortfall risk and is not plagued by inefficiencies, two conditions that the popular "4% rule" unfortunately violates.

Decumulation is a difficult problem, as put by two recipients of the Bank of Sweden Prize in economics: in 2019, Richard Thaler said that "decumulation is a more difficult challenge than accumulation", [2] which sounds almost as an echo to Bill Sharpe's emphatic view expressed a couple of years earlier that "decumulation is the nastiest, hardest problem in finance." [3] Indeed, the problem involves many sources of uncertainty, from market risks to longevity risk, and many control variables, which are withdrawal and investment decisions at any point in time. Moreover, bad investment and spending decisions may result in early depletion of savings, which may have dramatical consequences since many old individuals do not have the option to return to work to earn wages again.

Longevity risk can only be addressed by purchasing an annuity contract that provides lifetime income, but it cannot be hedged with purely financial securities. Because this paper restricts the investment universe to liquid securities traded in financial markets, like Government or investment-grade corporate bonds and listed equities, we leave aside this issue and consider a fixed horizon. Even after eliminating this source of complexity, the decumulation problem remains non-trivial. From the academic standpoint, Merton (1969, 1971, 1973) has derived an optimal consumption and investment policy for an agent who maximizes expected utility from future consumption and future bequest, and many subsequent papers have focused on the derivation of more explicit representations of the solution or on the relaxation of some certain assumptions. Among many other references, this literature includes the work of Kim and Omberg

[2] Thaler, R., 2019. Financial Advisors and Retirement: The Decumulation Dilemma. PIMCO Insight, October 28, 2019.
[3] Sharpe, W., 2017. Tackling the 'Nastiest, Hardest Problem in Finance'. Bloomberg Opinion, June 5, 2017.

(1996) and Wachter (2002), on portfolio optimization in the presence of a stochastic equity risk premium, and the work of Sangvinatsos and Wachter (2005) and Koijen, Nijman and Werker (2010), with a stochastic bond risk premium. Moving beyond Merton's time-separable utility function, Campbell (1993) and Campbell and Viceira (1999) solve the intertemporal planning problem with the recursive preferences of Duffie and Epstein (1992a, 1992b), which allow for the distinction between the risk tolerance and the elasticity of intertemporal substitution. While these results have provided invaluable advances in asset pricing, they are difficult if not impossible to use in practical situations at the personal finance level because all these optimal policies involve various subjective or objective parameters, such as investor risk aversion or expected returns on risky asset.

Another strand of the literature has a more applied objective, which is to guide financial advisors towards the choice of a fixed annual amount of spending to be recommended to their clients. The key parameter in such strategies is the initial withdrawal rate, which is the percentage of the nest egg that the retiree can consume in the first year and can continue to consume in the following years after making an adjustment for inflation. The historical simulation method of Bierwirth (1994) and Bengen (1994) leads Bengen to conclude that 4% is a safe rate. It follows from the work of Bengen (1994) and Cooley, Hubbard and Walz (1998) that such withdrawals have been feasible for at least 30 years in the data, at least for retirees holding 50% or more of their assets in US equities and the rest in bonds. While the 4% rule is straightforward to follow, Scott, Sharpe and Watson (2009) argue that it is "fundamentally flawed" because it imposes a constant dollar spending strategy upon a potentially volatile and risky investment strategy. In fact, a constant spending rule will inevitably lead to spend too little in certain scenarios and too much in others, so it cannot be safe and make an efficient use of savings at the same time. Even the safety of the 4% rule, which is the key property put forward by its proponents, is questionable, as explained by Pfau (2012), who notes that in many developed countries, historical data supports safe rates well under 4%.[4]

Scott, Sharpe and Watson (2009) note that to secure fixed replacement income for 30 years, an individual can simply purchase pure discount bonds with laddered maturities and spend the redemption values, a strategy that has no shortfall risk and never creates unspent surpluses. Other papers have made a case for the introduction of a similar risk-free asset in decumulation: Muralidhar, Ohashi and Shin (2016) call it "the most basic missing instrument in financial markets" or "bond for financial security", and Merton and Muralidhar (2017) introduce the closely related concept of "SeLFIES (Standard of Living indexed, Forward-starting, Income-only Securities)". Following Martellini, Milhau and Mulvey (2019), we call it a "retirement bond". This security plays a key role in the withdrawal and investment decisions analyzed in this paper. Indeed, we show that the retirement bond price, as implied by a market yield curve, is the input that serves to measure the purchasing power of savings in terms of replacement income, i.e., the amount of income that can be purchased with $1 of savings. The information about the retirement bond price thus allows us to revisit the problem of calculating a safe withdrawal rate and leads to a rate that is safe going forward as opposed to being predicated on the assumption that stocks and bonds will have similar performance as they had in the past.

While the strategy to purchase a retirement bond and to spend the coupons maximizes the replacement income subject to the constraint of never creating deficits, it may not be suitable for all individuals. Indeed, some of them may find that the safe withdrawal rate multiplied by their retirement savings is too low an amount to support the lifestyle that

[4] The lowest withdrawal rate sustainable over a 30-year period is that of Japan and is only 0.47%.

they aspire to, especially in a low interest rate environment that makes bonds expensive with respect to historical standards. For such retirees, the "purchasing power rule", which recommends that they spend an amount equal to the purchasing power of their initial savings every year, is no longer safe, so an alternative withdrawal policy is needed. We examine two such rules, which are safe whatever investment strategy is followed. The first one, called the "naïve annuitization rule", simply looks at the amount of savings every year and divides it by the number of forthcoming withdrawals. The second one is an extension of Siegel and Waring's (2015) "annually recalculated virtual annuity", which divides the amount of savings by the price of the retirement bond. Thus, the difference between the two is that the latter, which we call the "maximally moderate rule" for reasons explained in the paper, considers the time value of money through the discounting of future cash flows. Interestingly, that rule also coincides with Merton's optimal consumption policy for an infinitely risk averse investor. Taking the risk aversion parameter equal to infinity precisely removes the problematic dependence of the rule with respect to investment opportunities, except for interest rates, which are much more easily observable than risk premia.

The retirement bond price thus appears to be an important building block in the design of a withdrawal plan, and the bond itself has a role to play in retirees' portfolios, as the safe asset for those who want to secure fixed replacement income. We give a formal argument in favor of strategies combining the bond with a performance-seeking portfolio by deriving the optimal investment strategy for an agent following the maximally moderate rule in a general setting. Specializing this result to the case of constant investment opportunities, we provide a numerical illustration showing that the maximally moderate rule entails a much lower utility loss with respect to Merton's optimal policy than the naïve annuitization rule. Finally, we show that the substitution of the standard bond portfolio with the retirement bond, or a fixed-income portfolio replicating the retirement bond, in balanced funds and target date funds that are routinely used in retirement investing reduces downside risk in withdrawals. This result constitutes an encouraging sign that without implementing the optimal investment strategy, which depends on risk aversion and state variables like Merton's one, retirees can enjoy the benefits of the retirement bond even in simple strategies.

The rest of the paper is organized as follows. Section 1 offers a reminder of Merton's (1969, 1971, 1973) optimal consumption and investment policy. Section 2 introduces the retirement bond as an answer to the shortcomings of the 4% rule. In Section 3, we describe the naïve annuitization and the maximally moderate spending rules, and we provide an empirical illustration of their respective properties. In Section 4, we analyze portfolio strategies combining the retirement bond and other assets, including equities. We solve for the optimal investment policy for an individual who follows the maximally moderate rule, and we empirically compare balanced funds and target date funds using the retirement bond with funds invested in a traditional bond portfolio. Section 5 concludes.

1 Merton's Optimal Portfolio Strategy

We begin with a reminder of the optimal consumption and portfolio strategy from Merton's (1969) model. Merton's model provides a framework for the design of optimal decumulation strategies consistent with rationality assumptions about investors.

Starting with some initial wealth W_0, an agent maximizes the expected utility of future consumption by optimally choosing consumption c_t and the percentage weights in the various financial securities available, w_t, at each point in time. This optimization program is mathematically written as

$$\max_{w,c} E\left[\int_0^T e^{-\rho t} u(c_t) dt\right],$$

where u is the utility function, ρ is the subjective discount rate and T is the investment horizon.

In a dynamically complete market, there exists a unique stochastic discount factor M, which prices all securities. Munk and Sorensen (2004) show that in such a market, and under the assumption of constant relative risk aversion γ, the optimal portfolio on a given time t is a linear combination of the maximum Sharpe ratio portfolio of risky assets and a portfolio that (perfectly) replicates the returns on a coupon bond with price given by

$$Q_t = \int_t^T e^{-\frac{\rho}{\gamma}[s-t]} E_t\left[\left[\frac{M_s}{M_t}\right]^{1-\frac{1}{\gamma}}\right] ds. \tag{1}$$

(E_t denotes expectation conditional on the information available at time t.). In details, if Σ_t and μ_t denote the covariance matrix and the expected excess return vector of the assets, and $c_{Q,t}$ is the vector of covariances between asset returns and Q_t, we have

$$w_t^* = \frac{1}{\gamma} \Sigma_t^{-1} \mu_t + \Sigma_t^{-1} c_{Q,t}. \tag{2}$$

Munk and Sorensen (2004) also show that optimal consumption is given by

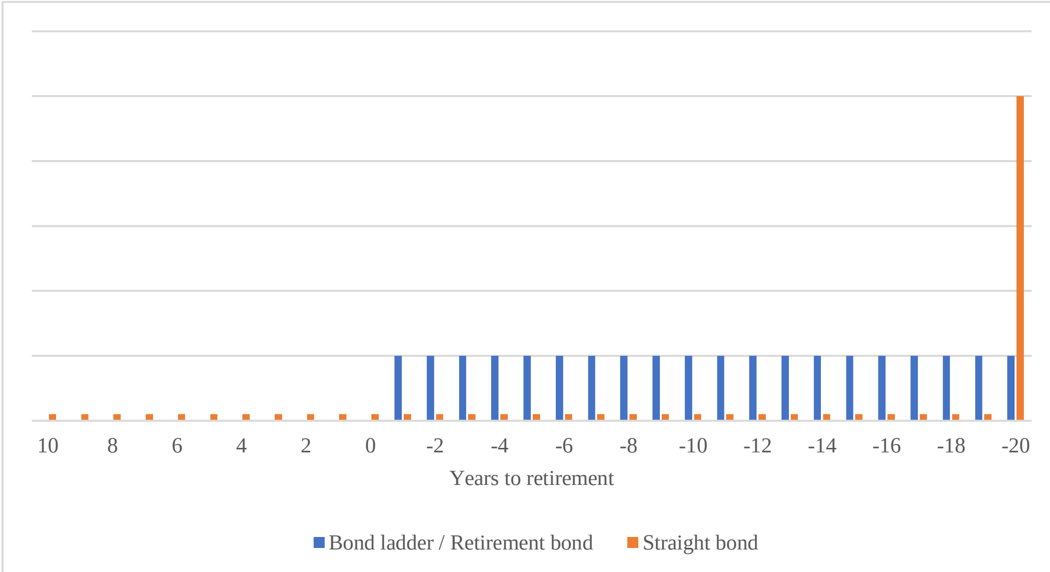
$$c_t^* = \frac{W_t}{Q_t}, \tag{3}$$

so Q_t is also the wealth-to-consumption ratio. A remarkable property is that this ratio is independent from current wealth but depends on investment opportunities at time t through the conditional expectation. Therefore, it is a function of current expected returns, volatilities and correlations.

At this level of generality, no explicit representation of Q_t can be obtained beyond Equation (1), but analytical formulas can be derived by specifying the dynamics of the state variables that drive investment opportunities. The literature provides a variety of such formulas; see e.g. Kim and Omberg (1996) and Wachter (2002) for mean-reverting equity risk premia, Munk and Sorensen (2004) for stochastic interest rates following the Vasicek model, and Liu (2007) for a generic framework encompassing "affine" models for state variables.

A common feature of these explicit solutions is that both the optimal portfolio and the consumption-to-wealth ratio depend on current investment opportunities and the parameters of the model followed by the state variables, e.g. the speeds of mean reversion, the volatilities, the long-term means and the correlations between all innovations to these variables. They are also functions of risk aversion. These dependencies make practical implementation problematic because risk aversion is unobservable and many state variables, including notably risk premia, are difficult to estimate. Therefore, we are not aware of any practical attempt to implement the optimal consumption and investment policy.

Figure 1: Cash flows of a bond ladder and a straight bond



Vertical scale (in dollars) is arbitrary.

The Infinite Risk Aversion Case

The infinite risk aversion case is interesting because dependence on many state variables vanishes. This can be seen by sending γ to infinity in Equation (1), which becomes

$$Q_t = \int_t^T E_t\left[\frac{M_s}{M_t}\right] ds.$$

It is the price of a coupon-paying bond that pays a continuous coupon of 1 per unit of time, hence a function of interest rates only. Among state variables, interest rates are the only ones that are observable, or quasi-observable up to the reconstitution of the yield curve from Government bond prices. Therefore, Q_t is much easier to calculate when risk aversion is infinite. Munk and Sorensen (2004) call the corresponding bond an "annuity bond".

It can be inferred from Equation (2) that the optimal portfolio policy in that limit case is to invest 100% of wealth in the bond-replicating portfolio. [5] So, the optimal strategy is to replicate a bond that pays a constant coupon of 1, which brings us back to a standard fixed-income portfolio management problem. The replicating portfolio can be constructed via duration-matching or duration-convexity-matching techniques.

[5] The work of Wachter (2003) provides a hint towards an extended version of this result beyond the case of constant relative risk aversion. Indeed, Wachter shows that for any utility function, the optimal portfolio for an investor who maximizes expected utility from terminal consumption only, and whose relative risk aversion grows to infinity, converges to a portfolio fully invested in a pure discount bond with maturity equal to the investment horizon. To the best of our knowledge, this result has not been proved for the case where utility depends on intermediate consumption.

2 Introducing the Retirement Bond as the Risk-Free Asset for Decumulation

A retirement bond is defined as a discrete-time version of the "annuity bond" arising in optimal consumption and investment planning. It is a security that pays constant cash flows (in nominal or cost-of-living-adjusted terms) on predetermined dates for a fixed period – typically for 15, 20 or 30 years after the retirement date. The introduction of this security as a useful asset for retirees can be justified by considering the case of infinite risk aversion in the optimization problem, but also by analyzing the limits of heuristic spending strategies that recommend a fixed withdrawal rate, as we explain below.

2.1 Cash Flow Schedule

Cash flows are normalized to $1, plus possibly a cost-of-living adjustment that is fixed for the bond's life and can be annual, monthly or any frequency. They begin at the retirement time, when an individual needs replacement income. This cash-flow schedule is different from that of a straight bond, which pays coupons beginning at the settlement time and redeems principal at maturity only, thereby creating an unbalanced series of payments. These two key differences are illustrated in Figure 1.

2.2 Motivation

A popular solution to the decumulation problem is to withdraw every year an amount equivalent to 4% of savings at the retirement time, plus realized inflation. The origins of the 4% rule are found in the work of Bierwirth (1994) and Bengen (1994). Bengen (1994) simulates spending rules using various percentages of initial savings, like 3%, 4%, 5% and 6%, over rolling decumulation periods of 50 years each. He goes on to find that with a portfolio consisting of 50% stocks and 50% bonds, 4% withdrawals adjusted for inflation every year were sustainable for 33 years at least, whatever the retirement year from 1926 to 1976, while a 5% rate would have exhausted savings in less than 25 years in some scenarios, e.g. if the individual had retired in the mid-1960s. Therefore, 4% is regarded as a compromise between a low rate, which preserves savings for a very long time at the cost of poor lifestyle, and a high rate, which allows for better lifestyle but is not sustainable over long periods.

The "Trinity study" of Cooley, Hubbard and Walz (1998) confirms these results over a slightly extended sample and adds important insights into the role of the investment strategy. Indeed, it turns out that 4% withdrawals adjusted for inflation have a 95% probability of being feasible over 30 years if savings are equally divided between stocks and bonds, and a 98% probability if the percentage allocation to stocks is 75%. But an individual who invests only in bonds has a success probability of 20% only (see their Table 3). This result questions the conventional wisdom that older investors, especially the retired ones, should favor bonds in their asset allocation. Historically, US bonds have delivered insufficient returns to support 4% withdrawals with an acceptable confidence level, but US stocks have.

However, Pfau (2010) warns that the 4% rate, which is derived from US data, might not be extrapolated to other countries. The safe withdrawal rate for a 30-year decumulation period with inflation adjustments estimated from 1900 - 2008 data was less than 4% in 13 out of 17 developed countries and was sometimes much less: France is found to have a rate of 1.25%, and Japan has the worst rate, at 0.47% only (see his Table 3). International data suggests that prudence is in order when extrapolating past rates not only to other countries, but also to the future. Dimson, Marsh and Staunton (2004) precisely criticize the "irrational optimism" of stock investors believing that future US stock returns will be as generous as they were in the 20^{th} century.

Whatever percentage is chosen, a problem inherent to "x%" rules is that they use the same rate regardless of the retirement time and economic conditions. The rate must be low enough to warrant feasibility in all scenarios, at least in those regarded as possible, but it will necessarily be too conservative in scenarios that are not "worst-case" periods. This observation concurs with Scott, Sharpe and Watson's (2009) point that "[s]upporting a constant spending plan using a volatile investment policy is fundamentally flawed". As they note, constant inflation-indexed income can be achieved by investing in a ladder of Treasury inflation-protected securities, a strategy that creates no risk of a deficit or a surplus. It is this bond ladder that we call the retirement bond.

A retirement bond provides fixed income, either in nominal or in real terms, for a fixed period, which would typically be the life expectancy at retirement time, e.g. 20 years for 65-year-old individual. Income in the late stage of retirement can be secured by purchasing a deferred annuity, which defers payments until the bond matures. If the individual dies before the bond maturity, the liquidity inherent to that asset ensures that it can be passed to heirs, unlike annuities, which cannot be transferred to another beneficiary. Liquidity also allows the bond owner to sell it if she needs money to fund a large upfront expense, like those generated by a life event. In contrast, the premia invested in an annuity are generally not recoverable.

2.3 Measuring the Purchasing Power in Terms of Replacement Income

Retirement Bond Pricing

The (dirty) price of a retirement bond is the sum of its discounted future cash flows. If cash flows are fixed in nominal terms, which encompasses the case where they include a constant cost-of-living adjustment, they must be discounted with nominal interest rates. If time 0 conventionally denotes the retirement time and π is the annual growth rate in the cost of living, the cash flow in year *t* is

$$CF_t = [1 + \pi]^t.$$

By letting $\pi = 0$, we obtain cash flows that are constant in dollars.

A more complex version of the retirement bond pays cash flows indexed on realized inflation. Then, the discount rates to be applied are *real* interest rates. To construct a bond-replicating portfolio for such an instrument, one needs to use inflation-indexed securities. Since indexed bonds form a less deep and liquid market than nominal bonds, we consider cash flows fixed in nominal terms in what follows.

Let 1, 2, …, *T* be the payment dates and assume continuous compounding of discount rates. Then, the bond price at time *t*, excluding the cash flow paid at that time, is given by

$$\beta_t = \sum_{\substack{s=1 \\ s>t}}^{T} CF_s\, exp\big[-[s-t]y_{t,s-t}\big],$$

where $y_{t,s-t}$ is the nominal zero-coupon yield prevailing at time t for a redemption horizon $s - t$. The bond price just before the cash-flow payment is

$$\beta_{t-} = \beta_t - CF_t.$$

Maximum Affordable Income

With the retirement bond price at hand, it is straightforward to measure the purchasing power of savings in terms of replacement income, that is the maximum number of cost-of-living-adjusted dollars that an individual can withdraw at all future dates. Formally, consider an individual endowed with wealth W_t at time t, and planning to make the following cost-of-living-adjusted dollar withdrawals on future withdrawal dates:

$$c_s = ri \times CF_s, \quad s > t.$$

ri denotes the real replacement income. The maximum possible value for ri, i.e., the maximum value that is compatible with the condition of a nonnegative wealth through time T, is

$$ri_{max,t} = \frac{W_t}{\beta_t}. \tag{4}$$

The only investment strategy that supports this maximal affordable income is 100% in the retirement bond, so any other strategy combined with these withdrawals will have positive probabilities of a deficit and of a surplus.

2.4 Empirical Illustration

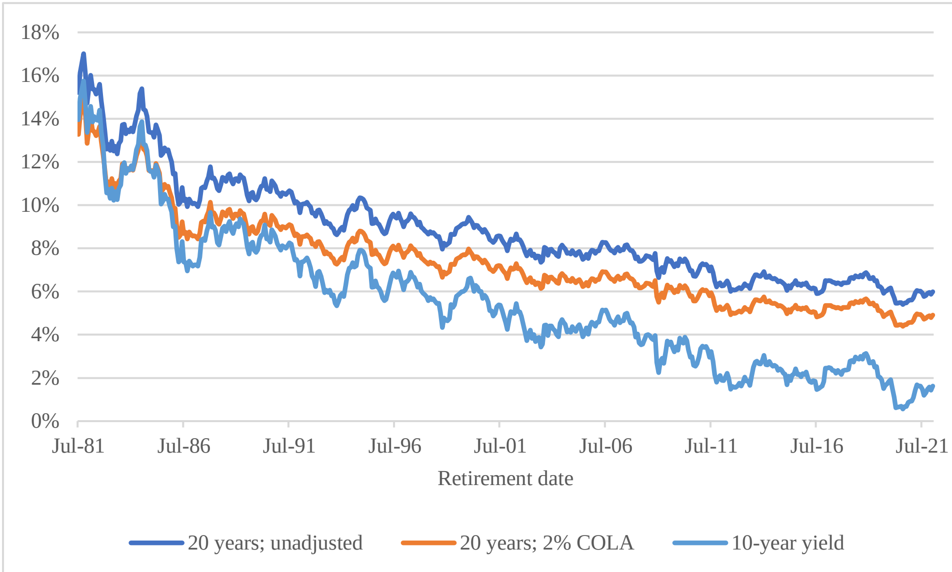
From Equation (4), the reciprocal of the retirement bond price at the retirement time, $1/\beta_0$, is the maximum withdrawal rate at that time. Because the bond price is increasing in the duration of the decumulation period and decreasing in the level of rates, the maximum withdrawal rate is decreasing in the duration and increasing in the level of rates.

Figure 2 displays the maximum withdrawal rates for a 20-year and a 30-year decumulation period, obtained by calculating retirement bond prices with US zero-coupon yields. The yields are borrowed from the dataset at Nasdaq Data Link,[6] which is based on the methodology of Gürkaynak, Sack and Wright (2007). Individuals retiring in July 1981 with a planning horizon of 20 years could withdraw 15.87% of their initial savings every year. If they planned to increase withdrawals by 2% per year, they had to reduce the rate to 13.27%, which corresponds to a nominal rate of $13.27\% \times 1.02 = 13.54\%$ in the first year and a rate of $13.27\% \times 1.02^{20} = 19.72\%$ in the last year. The situation

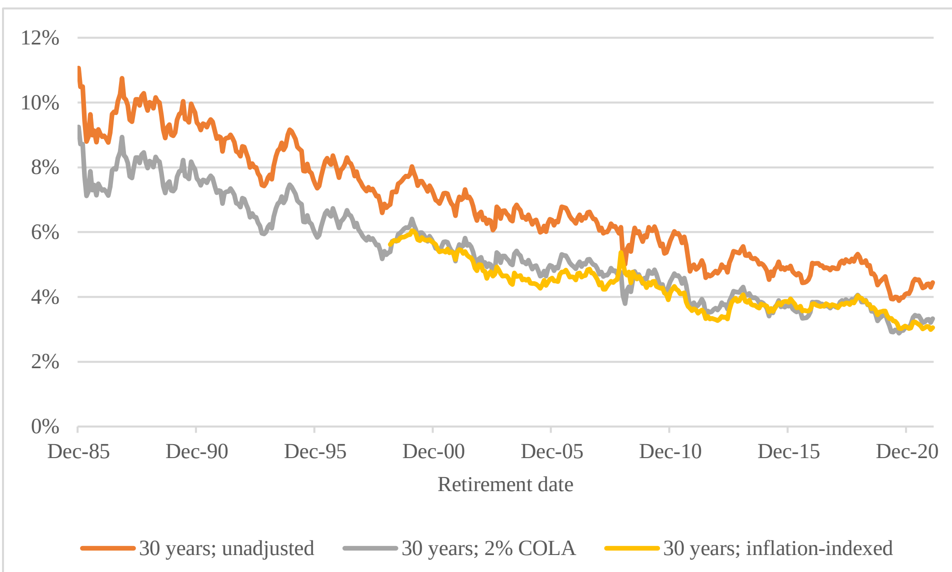
[6] The zero-coupon curves are available at https://data.nasdaq.com/data/FED/SVENY.

Figure 2: Maximal withdrawal rate.

(a) 20-year decumulation period.



(b) 30-year decumulation period.



The maximal withdrawal rate is the percentage of wealth at retirement that can be withdrawn every year for 20 or 30 years beginning one year after the retirement date. Annual withdrawals are constant in nominal terms – which corresponds to a 0% adjustment –, subject to a 2% annual adjustment for the cost of living or indexed on realized inflation. The 10-year yield is the market yield on US Treasury securities at 10-year constant maturity from the Federal Reserve.

was not as good for an individual retiring in January 2022, when interest rates were much lower. The yield on 10-year Treasury bonds was then 1.63%, while it used to be 13.95% in July 1981. Consequently, a 20-year retirement bond was much more expensive in 2022 than in 1981, and the withdrawal rate in January 2022 was only 6%, or 4.91% with a 2% COLA.

For a fairer comparison with the 4% rule, which has been tailored to a 30-year planning horizon with annual inflation adjustment, Figure 2(b) also displays the 30-year withdrawal rates without adjustment, with a 2% COLA and with indexation on realized inflation. The real zero-coupon yields are still obtained from Nasdaq Data Link and derived from the prices of inflation-protected securities by the methodology of Gürkaynak, Sack and Wright (2010).[7] The withdrawal rates with a 2% COLA and the inflation-adjusted rates are close to each other, suggesting that the breakeven inflation rate embedded in the nominal and real term structures is close to 2%, which is indeed the Federal Reserve's target inflation rate. The 4% prescription was clearly too conservative for retirees targeting constant income even in January 2022, when the rate was 4.44%. For individuals requiring inflation-adjusted income, the rule was also too conservative until August 2011, when the rate fell under 4%. In January 2022, these retirees have to live with a 3.05% rate.

3 Towards Feasible and Exhaustive Rules

One key insight from Figure 2 is that the maximal withdrawal rate depends on interest rates, so that any "x%" rule that recommends the same spending rate regardless of market conditions will lead to underspending in some scenarios and to overspending in others. In other words, such a rule is neither *feasible* nor *exhaustive*. *Feasibility* means that withdrawals can be made as planned without running into a negative balance of savings, and *exhaustivity* means that there is no residual money at the end of the decumulation period. Feasibility is obviously a desirable property because retirees do not want to run into liquidity issues, but exhaustivity is attractive too because it means that savings are efficiently spent, in the sense that no residual money is left.

3.1 "Purchasing Power" Rule

By investing her savings in the retirement bond, or a bond-replicating portfolio, an individual can follow the "purchasing power" spending rules, in which withdrawals are given by

$$c_t = ri_0 \times CF_t, \quad t = 1, \ldots, T.$$

This rule implies constant withdrawals (in real terms), but unlike "x%" rules, it does not prescribe the use of the same spending rate whatever the retirement time. We call it the "purchasing power" rule because it uses the purchasing power of savings as a key input.

The purchasing power rule combined with the retirement bond is feasible and exhaustive because it leads to a zero account balance after the last withdrawal in all scenarios. It is an improvement over the 4% rule, which enjoys neither of these properties, but it locks in the amount of replacement income. Individuals who find that the purchasing power of their savings is large enough to support the lifestyle that they want can follow that strategy, but in a context of low interest rates, chances are that the maximal withdrawal rate will be too low. The target replacement income is then non-affordable given the accumulated savings, so the strategies of fully investing savings in the retirement bond is not

[7] See https://data.nasdaq.com/data/FED/TIPSY.

advisable for these retirees. They could invest in a stock-bond fund, hoping that equity returns will increase the purchasing power of their savings, and follow the purchasing power rule until they reach their target, but this strategy carries a non-zero risk of depleting savings before time *T* and a non-zero (albeit less unpleasant) risk of a final surplus. They would then be back to the problems posed by the 4% rule. To avoid them, they need to pick another withdrawal policy, which warrants feasibility and exhaustivity for the investment policy that they follow.

3.2 "Naïve Annuitization" Rule

The "naïve annuitization" rule is inspired by the following observation. Just before the first withdrawal, at time 1, the individual has savings W_{1-} and expects to make T withdrawals now and in the future. If savings were to earn a zero rate of return every year, the maximum amount that could be withdrawn on every date would be W_{1-}/T. The first withdrawal is taken equal to that amount.

But savings are invested, so they earn a positive or negative return in the second year. Just before the second withdrawal, savings are W_{2-}. Repeating the calculation made at time 1, the individual finds that she can withdraw $W_{2-}/[T-1]$, since $T-1$ is the number of forthcoming withdrawals. The calculation is repeated every year, so that the withdrawal at time *t* is

$$c_t = \frac{W_{t-}}{T-t+1}, \quad t = 1, \ldots, T.$$

This strategy is clearly feasible because the withdrawal is always less than or equal to the account balance, and it is exhaustive too because the last withdrawal, equal to W_{T-}, exhausts any remaining savings. The unsatisfactory property of that rule is that it ignores discounting effects, reasoning as if savings were to earn no returns. The next rule that we introduce considers the time value of money.

3.3 "Maximally Moderate" Rule

The feasibility and exhaustivity conditions do not characterize a single withdrawal strategy, and they even leave the possibility of clearly unreasonable approaches, like those that would consume all savings in a given year and nothing in the other years. To exclude these extreme approaches, we propose to add a third condition that captures the notion of indifference between current and future consumption. This "moderation" condition is satisfied if the withdrawal made at time *t* is sustainable in the future in the sense that savings after the withdrawal are big enough to support the same withdrawal in the future. After incorporating the cost-of-living adjustment, this condition is mathematically written as

$$\frac{c_t}{CF_t} \leq \frac{W_t}{\beta_t}.$$

Let W_{t-} denote savings just before the withdrawal, so that

$$W_t = W_{t-} - c_t.$$

Substituting this expression and $\beta_t = \beta_{t-} - CF_t$ into the moderation condition and re-arranging terms, we obtain the equivalent condition

$$c_t \le \frac{W_{t-}}{\beta_{t-}} CF_t.$$

The right-hand side is the largest possible withdrawal compatible with the moderation condition.

The "maximally moderate" rule sets the withdrawal at time *t* to the maximum possible value, that is

$$c_t = \frac{W_{t-}}{\beta_{t-}} CF_t, \quad t = 1, \ldots, T.$$

Informally, this specification means that the individual favors neither current or future consumption over the other, since the purchasing power of savings immediately after the withdrawal is equal to c_t.

Like the naïve rule, the maximally moderate one is feasible and exhaustive in any scenario and for any investment strategy. Feasibility follows from the fact that $\beta_{t-} \ge CF_t$, and exhaustivity is warranted by the fact that $\beta_{T-} = CF_T$.

A similar rule was introduced by Siegel and Waring (2015) under the name "annually recalculated virtual annuity". The difference is that they divide wealth by the price of a TIPS bond ladder calculated with a discount rate equal to the historical average of real interest rates, while the maximally moderate rule uses market rates to calculate the bond price.

Finally, the maximally moderate rule is the discrete time equivalent of Merton's optimal consumption policy for an infinitely risk averse investor (see Equation (3)). Thus, it differs from the optimal rule in the assumption of infinite risk aversion for an individual who has some risk tolerance.

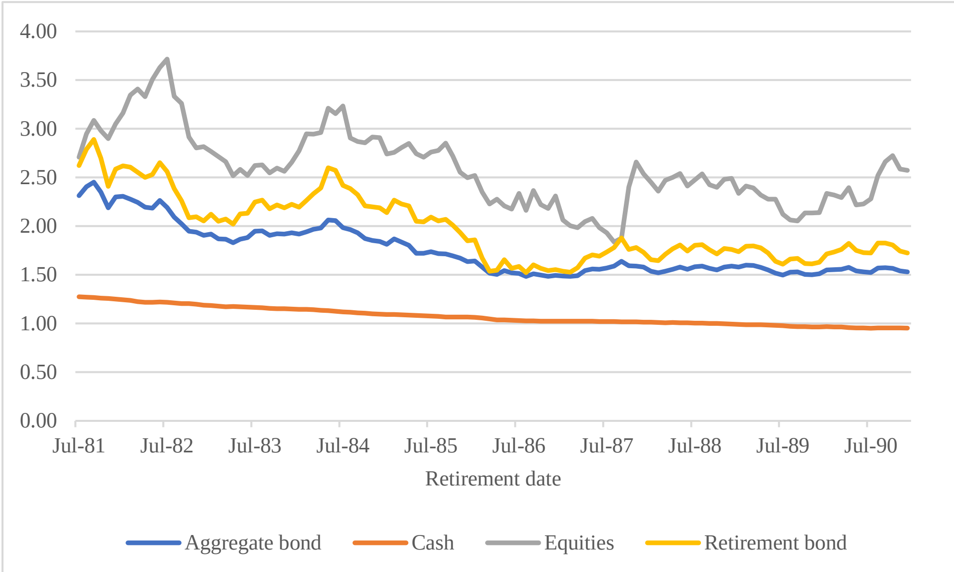
3.4 Empirical Illustration

To provide an empirical illustration of the properties of the various investment policies, we simulate them over decumulation periods with laddered start dates, following the methodology of Bengen (1994) and subsequent studies on the determination of safe withdrawal rates. The sample period from July 1981 to December 2020 is divided into overlapping decumulation periods of $T = 30$ years each, and we record withdrawals in each period. Because the withdrawals depend on the investment policy followed, we run the simulations for four hypothetical retirees, investing respectively in cash, the retirement bond, a Treasury bond index or equities.

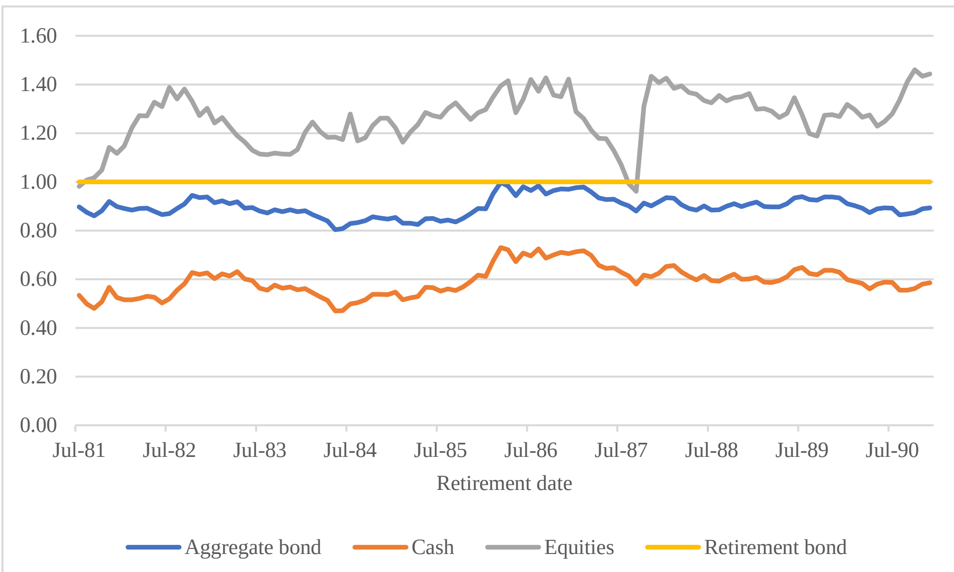
Cash is a monthly roll-over of US Treasury bills with a maturity of 3 months. Bonds are represented with the Bloomberg US Aggregate bond index, with coupons reinvested. This aggregate index contains Government and corporate bonds. The equity class is represented with the Scientific Beta US broad cap-weighted index, which is made with the 500 largest US stocks and has therefore close returns to those of the S&P 500 index. In addition to these standard classes, we also test the retirement bond, which, at this stage, is not an investable security until an investment-grade issuer offers that security or a fixed-income manager constructs a replicating portfolio. We do not apply a cost-of-living adjustment, so the bond cash flows are all equal to $1. Our sample begins in July 1981, when zero-coupon yields of maturities 16 years and longer start being available, and it ends in December 2020. With a dataset ending in December 2020, this procedure gives a total of 474 overlapping 30-year decumulation periods, but only those beginning from July 1981 to December 1990 are complete, while those beginning thereafter are incomplete to date.

Figure 3: Scaling factors of fixed spending rules.

(a) 4% rule.



(b) Purchasing power rule.



The scaling factor is the number by which all withdrawals must be multiplied for the final account balance to be zero. For each decumulation period, there are 30 annual withdrawals, which begin 12 months after the retirement date.

To simulate the 4% rule, we also use the US Consumer Price Index, to calculate withdrawals as

$$c_t = 0.04 \times W_0 \times \frac{CPI_t}{CPI_0}, \quad t = 1, \ldots T.$$

The other simulated rules are the purchasing power one, naïve annuitization and the maximally moderate one.

Feasibility and Exhaustivity of Withdrawal Strategies

To characterize the feasibility and the exhaustivity, or the lack thereof, of a given withdrawal strategy, we calculate an ex-post scaling factor, which is the factor by which all withdrawals should be multiplied to arrive at an account balance of zero. By construction, this factor is 1 if the strategy yields a zero final surplus in the first place, but it is less than 1 if the final surplus is negative, that is if the strategy is not feasible, and greater than 1 if it is not exhaustive.

The calculation of the scaling factor is based on the following budget equation, which expresses the final account balance, W_T, as a function of withdrawals and the returns on the underlying investment strategy. $R_{s,t}$ denotes the total gross return of the underlying investment strategy from time *s* to time *t*. We have

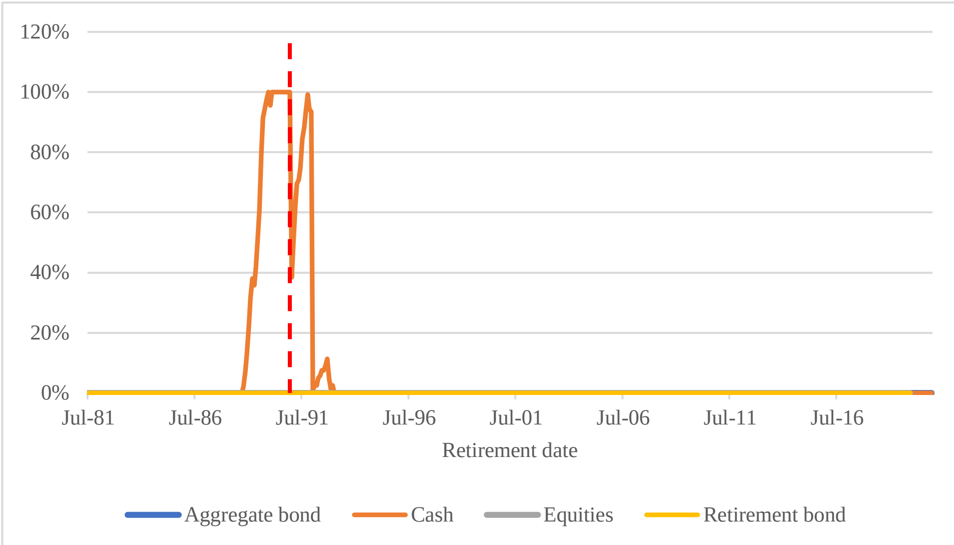
$$W_T = W_0 R_{0,T} - \sum_{t=1}^{T} c_t R_{t,T}. \tag{4}$$

The scaling factor *k* is defined by the following condition

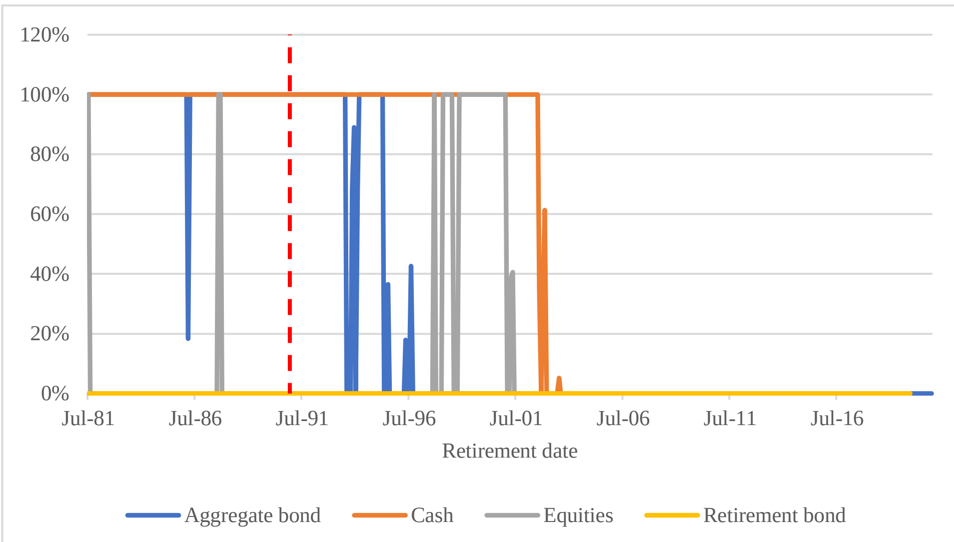
$$0 = W_0 R_{0,T} - k \sum_{t=1}^{T} c_t R_{t,T}.$$

Figure 4: Maximum drawdown in withdrawals.

(a) 4% rule.



(b) Purchasing power rule.



The maximum drawdown is recorded for every 30-year decumulation period. For every period, there are 30 annual withdrawals, which begin 12 months after the retirement date. The vertical dotted lines indicate the limit of complete decumulation periods, which begin until December 1990. For the 4% rule, the drawdown values are calculated from real withdrawals, i.e. withdrawals divided by the price index.

Substituting Equation (4) in this equation, it follows that

$$k = \frac{1}{1 - \frac{W_T}{W_0} \frac{1}{R_{0,T}}}.$$

To calculate k, we restrict to the period from July 1981 to December 1990 because the final account balance must be observed. As can be seen from Figure 3, the 4% rule had a considerable opportunity cost in our sample. With the aggregate bond, the factor ranged from 1.48 to 2.45 depending on when the individual retired, so withdrawals could have been uniformly raised by a percentage ranging from 48% to 145%. These numbers represent a missed opportunity to spend more, hence an opportunity cost. The larger performance of equities magnifies that cost, with a factor that ranges from 1.84 to 3.72. Owing to lower returns, investing in cash results in factors closer to 1, hence in a lower cost, but a feasibility problem arises: For individuals retiring from January 1989 on, the factor is less than 1, indicating that the 4% rule led them to overspend rather than to underspend.

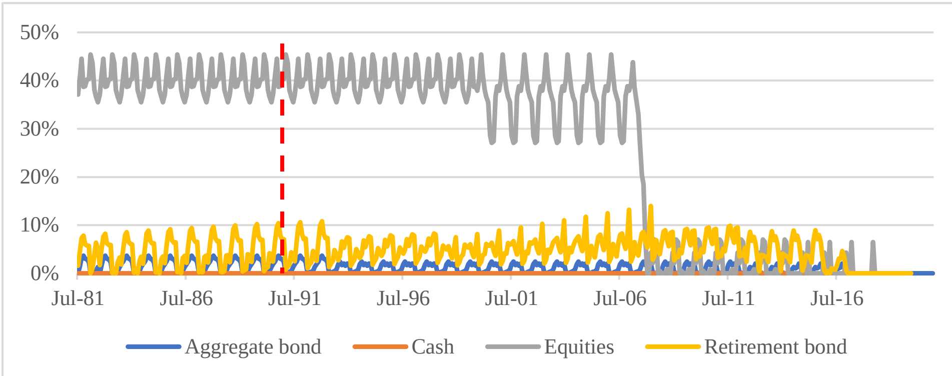
The fact that the 4% rule was always feasible with bonds in our sample seems to contradict the findings of Cooley, Hubbard and Walz (1998), who calculate that a 100% bond portfolio has only a 20% probability of supporting 4% withdrawals adjusted for inflation for 30 years (see their Table 3). These discrepancies can be attributed to the difference between the sample periods, since Cooley et al. work with returns from 1926 to 1995, while our bond return data spans the period from 1981 to 2020. It turns out that bonds delivered higher returns in the latter period because of decreasing interest rates. Dimson, Marsh and Staunton (2002) report that bonds and bills earned on average 1.9% and 1.0% per year between 1920 and 1989, versus 7.2% and 3.6% from 1980 to 1989. Extending the sample period to 2000 leads to similar differences: annual returns on bonds and bills were 2.6% and 1.1% from 1920 to 2000, and 6.9% and 2.8% from 1980 to 2000. These figures suggest that the scenarios where the 4% rule has failed in the results of Cooley et al. belong to the pre-1980 period, which is not present in our data.

Drawdown Risk in Withdrawals

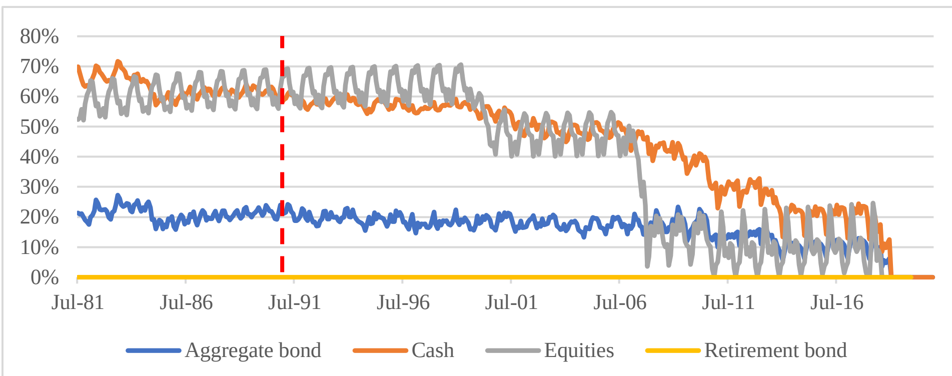
Maximum drawdown in withdrawals is a good measure of the downside risk in a sequence of withdrawals because it represents the magnitude of the maximum cut in replacement income. In theory, fixed spending rules like the 4% and

Figure 5: Maximum drawdown in withdrawals (con't).

(a) Naïve annuitization rule.



(b) Maximally moderate rule.



The maximum drawdown is recorded for every 30-year decumulation period. For every period, there are 30 annual withdrawals, which begin 12 months after the retirement date. The vertical dotted lines indicate the limit of complete decumulation periods, which begin until December 1990.

the purchasing power ones have no drawdown risk because withdrawals are fixed, but the liquidity constraint that all individuals face creates such risk. Specifically, no spending cut takes place while withdrawals can be financed, but savings fall to zero if savings are too low to support the scheduled withdrawal. If savings are exhausted at some point in a given period, the maximum drawdown in that period is positive, and it is 100% if depletion occurs before the last year.

The results in Figure 4 confirm that the 4% and the purchasing power rules involve drawdown risk and that the retirement bond is the only asset that supports withdrawals equal to the purchasing power of savings without causing a deficit. Figure 5 shows a cyclical pattern in the maximum drawdowns of the naïve annuitization and the purchasing power rules. For instance, every individual retiring in February from 1982 to 2006, investing in equities and making the naïve withdrawals experienced a maximum drawdown of 45.41%, regardless of the retirement year. An equity investor following the maximally moderate rule faced worse maximum drawdowns if she retired in March than in any other month, and the values at a one-year distance are close to each other: The individual retiring in March 1982 experienced a maximum drawdown of 64.85%, and the one retiring one year later got a slightly greater number, at 65.67%.

A remarkable property is that although the naïve annuitization and the maximally moderate rules are equivalent in terms of feasibility and exhaustivity, they imply very different levels of drawdown risk. For a cash investor, the former rule involves no drawdown risk, meaning that withdrawals are non-decreasing over time, but the latter rule results in drawdowns ranging from 57.14% to 71.70%. Similarly, for a bond investor, drawdowns are greater under the maximally moderate rule than under the naïve rule, although the spread between the two rules is not as large. The only asset for which maximum drawdown is lower under the maximally moderate policy is the retirement bond, since investing in that asset leads to constant withdrawals, equal to the purchasing power of initial savings.

3.5 A Formula for Naïve and Maximally Moderate Withdrawals

The key to understanding the results on maximum drawdown is the following proposition, the proof of which is written in Appendix A1.

Proposition 1. *Let $R_{s,t}$ denote the total gross return of the underlying investment strategy from time s to time t, and $\rho_{s,t}$ denote the total return of the retirement bond over the same period. Then, withdrawals under the naïve annuitization rule are given by*

$$c_t = \frac{W_0}{T} R_{0,t}, \quad \text{for } t = 1, \dots, T,$$

and withdrawals under the maximally moderate rule are given by

$$c_t = ri_{\max,0} \frac{R_{0,t}}{\rho_{0,t}}, \quad \text{for } t = 1, \dots, T.$$

The first equality in Proposition 1 says that naïve withdrawals are proportional to the cumulative return of the underlying investment strategy since retirement time, which implies that the drawdown in withdrawals equals the drawdown in the underlying fund. If equity values are sampled at the annual frequency on February ends, the worst drawdown is recorded from February 2007 to February 2009 and is 45.41%. It is exactly the height of the peaks in Figure 5b, and the drawdown recorded for individuals who retired in February from 1982 to 2006. All these retirees had to go through the bear market of 2008, so they experienced the same maximum relative cut in spending in that period. The same reasoning can be done for other months.

Proposition 1 also explains why drawdown risk under the naïve rule is absent for cash investors: A cash account has non-decreasing value, so its drawdown is always zero. For an aggregate bond investor, the maximum drawdown in withdrawals equals the maximum drawdown of the bond, which is much lower than that of equities: In the worst period, it is 3.67%, while it reaches 45.41% for equities.

For the maximally moderate rule, spending is proportional to the relative returns of the fund with respect to the retirement bond, so what drives uncertainty over spending is the relative risk of the fund, not its absolute risk. This makes a substantial difference with respect to the naïve policy because relative risk can be very different from absolute risk. The most obvious example is with the retirement bond. The relative risk of that strategy is zero, so the maximum drawdown in maximally moderate withdrawals is zero, but the retirement bond involves capital loss risk, owing to its long duration. As a matter of fact, a retiree investing in that bond and spending naively experienced a relative cut in spending of 13.67% at most. Cash is another asset for which the two risks differ. Absolute drawdown risk is zero, but relative drawdown risk is large, so maximally moderate investors had to bear reduction in spending up to 71.70%. These figures confirm the Merton's (2014) point that the low absolute risk of cash does not make it a safe asset for generating replacement income.

4 Using the Retirement Bond in Multi-Asset Strategies

Even though the retirement bond is the safe asset for decumulation, the strategy to invest all savings in that security is not to be recommended to every individual, precisely because the safe asset eliminates any downside risk in withdrawals but also removes upside potential. Proposition 1 shows that under the maximally moderate spending rule, withdrawals increase with respect to the initial purchasing power of savings if the portfolio in which savings are invested outperforms the retirement bond. This result provides motivation for investing in a performance-seeking portfolio (PSP) with a higher expected return than the bond. But the concern over downside risk calls for investing in the retirement bond too, so as to avoid severe spending cuts in a bear market. Therefore, retirees who want to earn more replacement income than what their initial savings can finance have reasons to invest both in a PSP and the retirement bond – or a replicating portfolio thereof.

To provide a more formal argument in favor of such "multi-asset strategies", we first derive the optimal investment policy under the maximally moderate rule. Like Merton's optimal portfolio strategy, it depends on state variables and risk aversion, so it would be difficult to implement, so we empirically study strategies that do not involve unobservable parameters. Because balanced funds and target date funds are popular in retirement investing, they provide a good starting point. We present a series of empirical results showing that substituting the standard bond portfolio of these funds with the retirement bond improves the risk-return profile of the replacement income that they deliver.

4.1 Theoretical Motivation: A Fund Separation Result

Economy

Following Merton (1969, 1971, 1973), we cast the portfolio optimization problem in a continuous-time setting. The time span is $[0, T]$, and uncertainty is represented with a d-dimensional Brownian motion *z*. At time *t*, the agent has wealth W_t and chooses how much to consume in period $[t, t + dt]$. Savings are invested in a fund whose total return index is denoted with X_t. The fund is invested in cash and *N* risky assets that have expected returns μ_t and unexpected returns $\sigma_t' dz_t$, where σ_t is a $d \times N$ matrix of loadings. For the sake of generality, we allow for a greater number of sources of risk than the number of assets, so the market may be incomplete.

The fund weights in the risky assets at time *t* are denoted with w_t and the short-term risk-free rate with r_t, so we have the following budget constraints:

$$dW_t = W_t \frac{dX_t}{X_t} - \frac{W_t}{\beta_t} dt,$$

$$\frac{dX_t}{X_t} = r_t dt + w_t' \mu_t dt + w_t' \sigma_t' dz_t.$$

The instantaneous covariance matrix of the risky asset returns is $\Sigma_t = \sigma_t' \sigma_t$.

The retirement bond pays a continuous coupon dt in $[t, t + dt]$ and has price β_t. Unlike in Merton's model, we do not treat consumption as a separate control variable because we assume that it is given by the maximally moderate rule, that is

$$c_t = \frac{W_t}{\beta_t},$$

where β_t denotes the price of a retirement bond that pays a continuous coupon.

Investment opportunities are functions of a vector of *K* state variables Y_t, which follow the vector diffusion process

$$dY_t = m_{Y,t} dt + \sigma_{Y,t}' dz_t.$$

The vector $m_{Y,t}$ has length *K*, and the loading matrix $\sigma_{Y,t}$ has shape $d \times K$. The matrix of covariances between the risky asset returns and the state variables is $c_{Y,t} = \sigma_t' \sigma_{Y,t}$.

Let $\widetilde{\beta_t}$ denote the total return index on the retirement bond. The total return in the period $[t, t + dt]$ is

$$\frac{d\widetilde{\beta_t}}{\widetilde{\beta_t}} = \frac{d\beta_t + dt}{\beta_t}.$$

By applying Ito's lemma, it can be shown that the result of Proposition 1 extends to the continuous-time setting, that is

$$c_t = \frac{W_0}{\beta_0} \frac{X_t}{X_0} \frac{\widetilde{\beta_0}}{\widetilde{\beta_t}}.$$

The total return index follows a diffusion process of the form

$$\frac{d\widetilde{\beta_t}}{\widetilde{\beta_t}} = \left[r_t + \mu_{\beta,t}\right] dt + \sigma_{\beta_t}' dz_t,$$

so the vector of covariances between the risky assets and the retirement bond is $c_{\beta,t} = \sigma_t' \sigma_{\beta,t}$.

Optimization Problem

The agent chooses the investment strategy so as to maximize the expected utility of future consumption:

$$\max_w E\left[\int_0^T u(c_t) dt\right]. \qquad (*)$$

Final wealth does not contribute to welfare because it is zero.

We assume that the investor has constant relative risk aversion γ, so that

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}.$$

Then, the optimization program is equivalent to

$$\max_w E\left[\int_0^T u\left(\frac{W_0}{\beta_0}\frac{X_t}{X_0}\frac{\widetilde{\beta_0}}{\widetilde{\beta_t}}\right)dt\right].$$

With the CRRA utility function, the ratio $W_0\widetilde{\beta_0}/[\beta_0 X_0]$ can be factored out of the integral and the expectation, so it is irrelevant to the maximization program. Therefore, indirect utility is a function of time, the relative fund value $\widehat{X_t} = X_t/\widetilde{\beta}_t$ and the state variables:

$$J\left(t, \widehat{X_t}, Y_t\right) = \left[\frac{W_0}{\beta_0 \widehat{X_0}}\right]^{1-\gamma} \times \max_{(w_s)_{s \geq t}} E_t\left[\int_t^T u\left(\frac{X_s}{\widetilde{\beta_s}}\right)ds\right].$$

Fund Separation Theorem

The following proposition, which is proved in Appendix A2, provides a representation of the optimal portfolio strategy for an investor following the maximally moderate spending rule.

Proposition 2. *The optimal vector of weights in the risky assets at time t for Program (*) is*

$$w_t^* = \frac{1}{\gamma}\Sigma_t^{-1}\mu_t + \left[1 - \frac{1}{\gamma}\right]\Sigma_t^{-1}c_{\beta,t} + \frac{1}{\gamma G}\Sigma_t^{-1}c_{Y,t}G_Y,$$

where G_Y is the gradient vector of the function G of time and state variables such that

$$J\left(t, \widehat{X_t}, Y_t\right) = \left[\frac{W_0}{\beta_0 \widehat{X_0}}\right]^{1-\gamma} u\left(\widehat{X_t}\right)G(t, Y_t).$$

Proposition 4 shows that the optimal portfolio strategy involves *K* + 2 risky funds. The first is the tangency portfolio, which achieves the highest short-term Sharpe ratio. The second is the portfolio that is most correlated with the returns on the retirement bond. If the market is complete, that portfolio perfectly replicates the retirement bond returns. The next *K* funds are hedging portfolios against unexpected changes in the state variables, like in Merton (1973), and those risky funds are completed with the cash account.

Let us compare the optimal portfolio of Proposition 2 with Merton's optimal portfolio. Liu (2007) shows that the optimal portfolio obtained by jointly optimizing over consumption and investment is[8]

$$w_t^{**} = \frac{1}{\gamma}\Sigma_t^{-1}\mu_t + \frac{1}{\gamma H}\Sigma_t^{-1}c_{Y,t}H_Y,$$

[8] This equation is obtained by rewriting Equation (6) from Liu (2007) with a few notational changes. Our function *H* corresponds to f^{γ} in Liu's notations.

where the value function is $I(t, W_t, Y_t) = u(W_t)H(t, Y_t)$. An obvious difference between the two portfolios that the one in Proposition 2 involves a third building block, which is the retirement bond-hedging portfolio. The rationale for holding a long position (when $\gamma > 1$) in that additional fund is that consumption is a decreasing function of the retirement bond price. Indeed, an unexpected rise in the bond price implies an unexpected cut in spending, unless it is compensated by an increase in the fund value. The extent to which the investor is willing to accept mismatch between the respective returns of the fund and the retirement bond depends on her risk tolerance, and the fraction of wealth invested in the retirement bond-hedging portfolio is an increasing function of risk aversion.

4.2 Welfare Loss with Respect to Merton's Optimal Strategy

By optimizing only over the investment rule, an individual incurs a welfare loss with respect to Merton's optimal consumption and investment plan. In this section, we quantitatively measure this loss in the case of constant parameters, which allows for an explicit solution of both optimization problems, and we compare it with the loss associated with the naïve annuitization rule.

Certain Equivalent Wealth and Consumption

We convert indirect utility into a "certain equivalent wealth", defined as the certain wealth W_{eq} that makes the individual indifferent between receiving W_{eq} today and investing W_0 in the optimal strategy and obtaining random consumption or wealth. For the maximally moderate rule and Merton's rule, the certain equivalent wealth is defined as

$$u\left(W_{eq,mm}\right) = \left[\frac{W_0}{\beta_0 \widehat{X_0}}\right]^{1-\gamma} J\left(0, \widehat{X_0}, Y_0\right),$$

$$u\left(W_{eq,me}\right) = I(0, W_0, Y_0).$$

Because $J\left(0, \widehat{X_0}, Y_0\right) = u\left(\widehat{X_0}\right) G(0, Y_0)$ and $I(0, W_0, Y_0) = u(W_0)H(0, Y_0)$, we obtain

$$W_{eq,mm} = \frac{W_0}{\beta_0} G(0, Y_0)^{\frac{1}{1-\gamma}},$$

$$W_{eq,me} = W_0 \, H(0, Y_0)^{\frac{1}{1-\gamma}}.$$

With constant investment opportunities, we have the following expressions for certain equivalent wealths under the Merton's, the maximally moderate and the naïve annuitization rules (see Appendix A4):

$$W_{eq,me} = W_0 \left[\frac{1 - exp[-a_{me} T]}{a_{me}}\right]^{\frac{\gamma}{1-\gamma}}, \qquad a_{me} = -\frac{1-\gamma}{\gamma}\left[r + \frac{\lambda_{MSR}^2}{2\gamma}\right],$$

$$W_{eq,mm} = \frac{W_0}{\beta_0}\left[\frac{1 - exp[-a_{mm} T]}{a_{mm}}\right]^{\frac{1}{1-\gamma}}, \qquad a_{mm} = -\frac{1-\gamma}{2\gamma}\lambda_{MSR}^2,$$

$$W_{eq,na} = \frac{W_0}{T}\left[\frac{1-exp[-a_{na}T]}{a_{na}}\right]^{\frac{1}{1-\gamma}}, \qquad a_{na} = -[1-\gamma]\left[r + \frac{\lambda_{MSR}^2}{2\gamma}\right].$$

With a constant interest rate, the retirement bond price is given by

$$\beta_0 = \frac{1-exp\big[-r[T-t]\big]}{r}.$$

When preferences are expressed over intermediate consumption rather than final wealth, it is more natural to express indirect utility in terms of consumption than as an equivalent wealth. To this end, we introduce the "certain equivalent consumption", defined as the certain and constant amount of consumption that implies the same expected utility as the optimal strategy. This quantity is related to the certain equivalent wealth through

$$\int_0^T u\big(c_{eq}\big)dt = u\big(W_{eq}\big),$$

hence

$$c_{eq} = T^{\frac{1}{1-\gamma}} W_{eq}.$$

Both the certain equivalent consumption and the certain equivalent wealth are in dollars and are proportional to W_0. To have a dimensionless quantity independent from initial wealth, we divide c_{eq} by the constant amount of consumption that would exhaust savings if savings were not invested and were thus to earn a zero rate of return in all periods. That constant amount is

$$c_{ct} = \frac{W_0}{T},$$

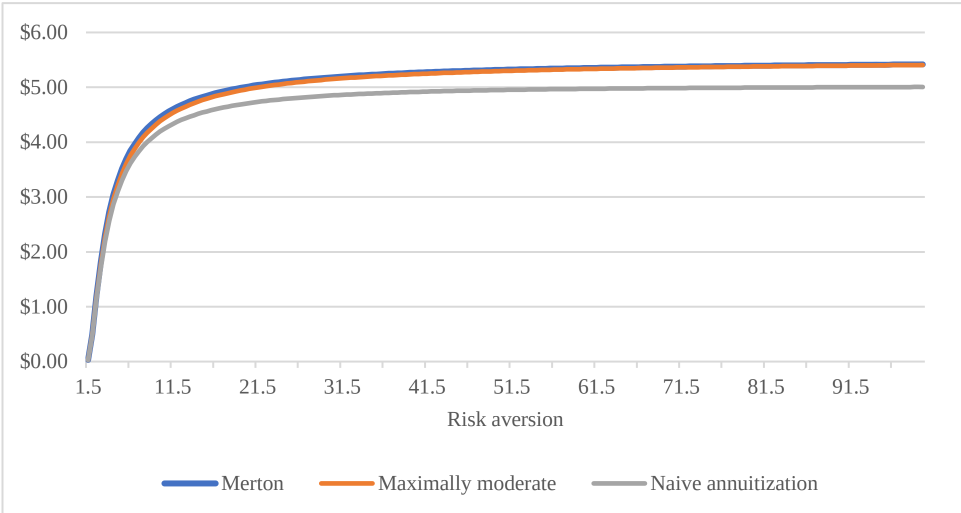
so we calculate the ratio c_{eq}/c_{ct}, which is dimensionless and independent from initial savings.

Numerical Illustration

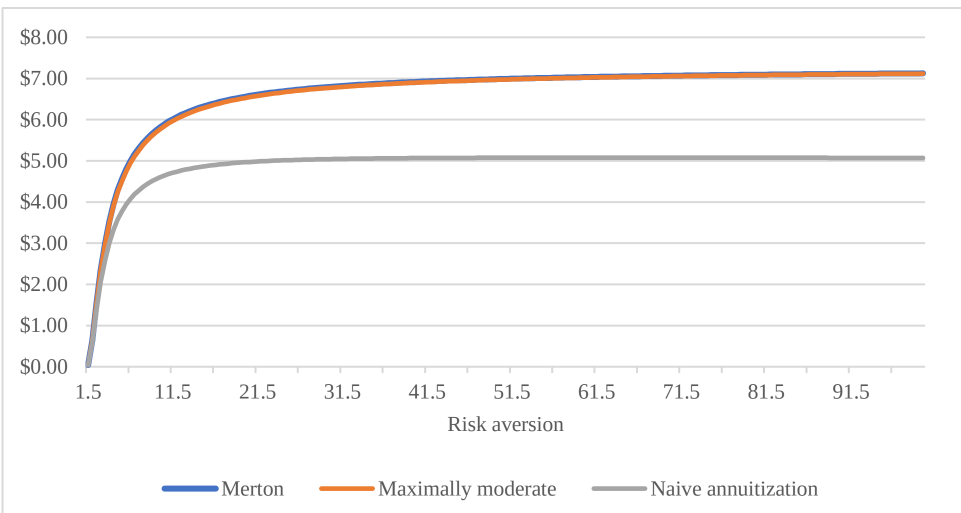
We set the investment horizon to 20 years, the maximum Sharpe ratio to 0.5, the short-term rate to 1% or 4%, and we let the risk aversion parameter vary from 1.5 to 100. Figure 6 shows the certain equivalent wealths. By construction, Merton's rule dominates any other rule, but a striking feature is that the welfare loss caused by the use of the maximally

Figure 6: Certain equivalent wealth per dollar of invested for various consumption rules under the optimal investment strategy.

(a) 1% interest rate.



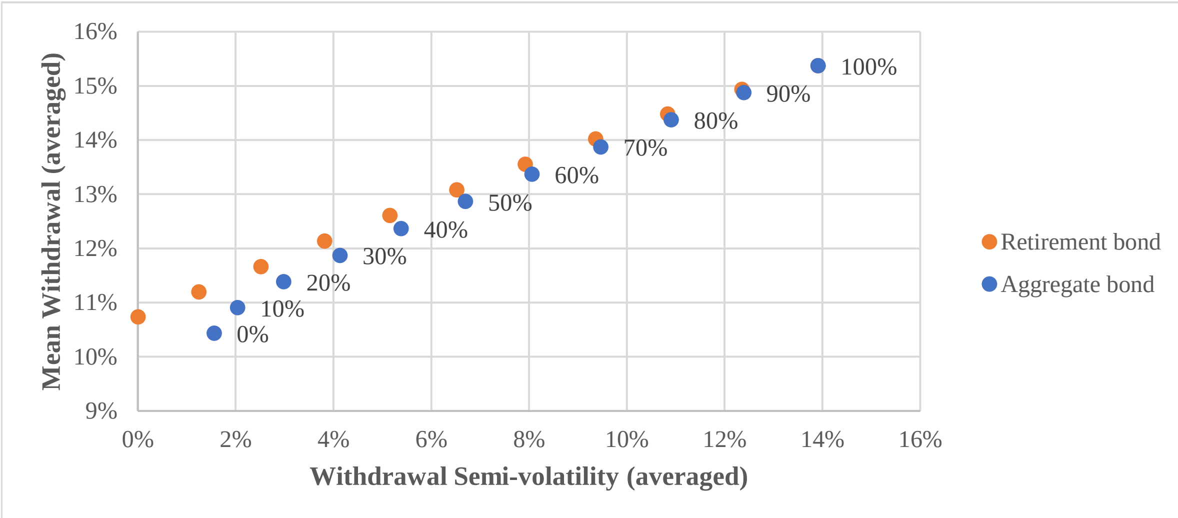
(b) 4% interest rate.



The certain equivalent wealth is the wealth that gives the same expected utility as investing $1 and following Merton's consumption rule, the maximally moderate rule or the naïve annuitization rule. The investment strategy is the optimal policy for all these consumption rule, which has weights $w^* = \Sigma^{-1}\mu/\gamma$ *in the risky assets and the rest in cash.*

moderate rule is very limited and much lower than the loss associated with the naïve annuitization rule. The naïve rule can be regarded as a special case of the maximally moderate rule in which discount rates are taken equal to zero, so that the retirement bond price is numerically equal to the remaining investment horizon. Therefore, it is not surprising that the distance between that rule and the maximally moderate one increases when the short-term rate grows from 1% to 4%.

Figure 7: Substitution of the aggregate bond with the retirement bond in balanced funds.



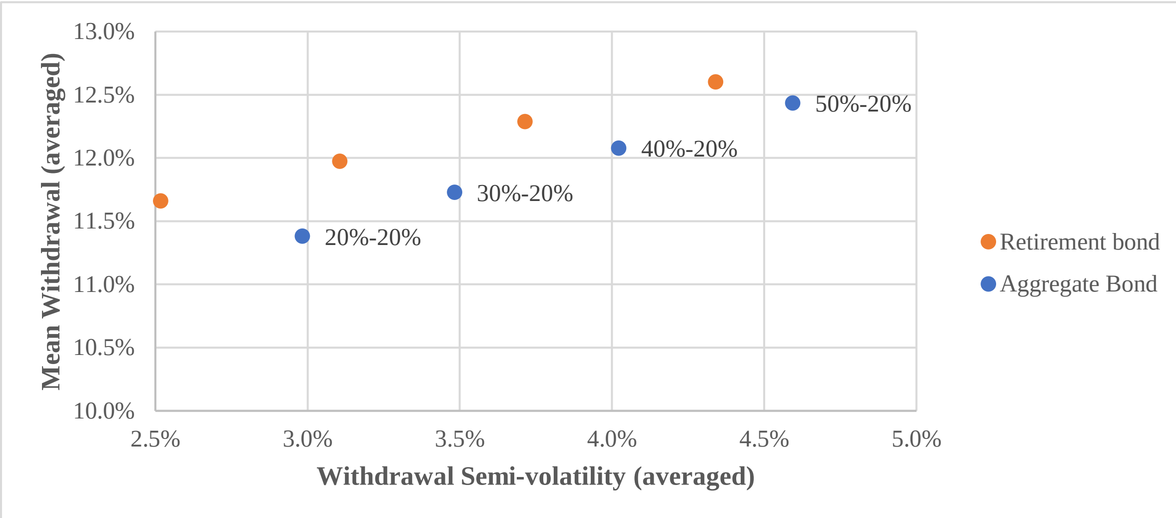
For each equity allocation ranging from 0% to 100% and each twenty-year decumulation period, the account balance for an individual who follows the maximally moderate withdrawal rule and invests in a balanced fund is simulated, and i) the mean withdrawal (expressed as a percentage of initial savings), and ii) the semi-volatility of log changes in withdrawals from one year to the next, are recorded. With the 234 decumulation periods, two distributions of 234 mean and semi-volatility values are obtained, for which the respective averages are reported.

4.3 Improving Balanced Funds and Target Date Funds

Balanced and target-date funds are invested in stocks and bonds, the former playing the role of a performance-seeking asset and the latter being viewed as a safe asset. But despite the conventional view that bonds are a more appropriate asset class than equities for retired investors, owing to their lower volatility and lower downside risk, the bond bucket of standard funds is not safe for the objective to produce stable replacement income. Indeed, the asset that makes fixed payments throughout retirement is the retirement bond, and it has decreasing duration over time, unlike the bond buckets of standard funds.

The duration mismatch in the bond bucket of target date funds is illustrated by Bilsen, Boelaars and Bovenberg (2019), who document that the actual durations are relatively insensitive to the investor's age, and if at all, they are not longer at younger ages (see their Figure 1). This property is inconsistent with a key property of the duration of a stream of constant payments, which decreases over time – and is thus decreasing in age – because of the maturity effect. Ironically, the authors point that the combination of a stable duration in the bond bucket and an increasing bond weight over time produces an increasing duration at the fund level, while the optimal interest rate exposure, as predicted by intertemporal portfolio optimization models, should be decreasing over time (see Proposition 1 and Theorem 4 in Brennan and Xia (2002) and Proposition 1 in Munk, Sorensen and Vinther (2004). Indeed, according to fund separation principles, an agent who maximizes expected utility from future consumption must invest part of her wealth in a bond that supports constant expenditure, and that bond has decreasing duration. Target-date funds allocate a non-decreasing

Figure 8: Substitution of the aggregate bond with the retirement bond in target date funds.



For each glidepath and each twenty-year decumulation period, the account balance for an individual who follows the maximally moderate withdrawal rule and invests in a target date fund is simulated, and i) the mean withdrawal (expressed as a percentage of initial savings), and ii) the semi-volatility of log changes in withdrawals from one year to the next, are recorded. With the 234 decumulation periods, two distributions of 234 mean and semi-volatility values are obtained, for which the respective averages are reported.

weight to a bond portfolio with a stable duration, which Bilsen, Boelaars and Bovenberg (2019) refer to as the "duration puzzle".

In a balanced fund, the percentage allocation to stocks and bonds is constant over time, and like in target date funds, the bond portfolio is not designed to produced truly stable replacement income. The modified versions of these funds that we analyze in this section are obtained by replacing the bond bucket with a portfolio assumed to perfectly replicate the retirement bond. We compare them with standard funds invested in equities and the aggregate bond index by simulating them over 234 decumulation periods of 20 years each, beginning each month from July 1987 to December 2000. In each period, we record the mean withdrawal expressed as a percentage of initial savings and the semi-volatility of annual logarithmic changes in withdrawals, and we finally average the 234 performance and risk statistics across periods. Semi-volatility, calculated as the square root of the partial moment of order 2 restricted to negative annual changes in income, is preferred to volatility because it penalizes downside risk.

By repeating the calculations for balanced funds with equity allocations ranging from 0% to 100%, we obtain a series of mean replacement income and a series of (averaged) income semi-volatility, which are plotted in Figure 7 in the form of a pseudo-efficient frontier. Like the usual mean-variance frontier, it features a "minimum risk allocation", which corresponds to 0% in equities on average but not necessarily for every decumulation period: for approximately 27% of the decumulation periods in our sample, the minimum risk allocation is obtained with a non-zero exposure to equities (typically between 10 and 20%) when the bond portfolio is the standard aggregate bond. On the other hand, the minimum semi-volatility is attained and equal to zero with a 0% equities allocation in every decumulation period when the bond portfolio is the retirement bond.

The key message from Figure 7 is that the substitution of the commonly used aggregate bond with the retirement bond shifts the frontier to the north west, which represents an improvement. For every equity allocation, the modified fund is less risky than the standard one and overall produces a higher income. For example, the average semi-volatility for a 30%-equity fund decreases from 4.13% to 3.82% while the average mean withdrawal increases from 11.87% to 12.13%, so the terms of the risk-return tradeoff improve when the substitution is performed. Retired investors may take advantage of the substitution in two ways. Either they can reduce risk while keeping the same allocation to equities, or they can allocate more to equities, thereby increasing the upside potential of their portfolio, without increasing the semi-volatility of their replacement income.

Figure 8 provides a similar graphic representation for target date funds, with four glidepaths. They start out respectively with 20%, 30%, 40% and 50% in equities at the retirement time and end with 20% after 20 years. Like in balanced funds, the substitution reduces semi-volatility in replacement income. For example, the 40-20 fund invested in equities and the aggregate bond leads to an average semi-volatility of 4.02% combined and an average mean withdrawal of 12.08%, while the fund with the same glidepath invested in the retirement bond leads to metrics of 3.71% and 12.29%.

5 Conclusion

Decumulation, which is the process of turning retirement savings into replacement income, is a difficult task if one wants to avoid the pitfalls of depleting savings before the planning horizon and being exceedingly parsimonious in consumption. Optimal consumption and investment policies derived by maximizing a utility function are both feasible and exhaustive, i.e., they avoid both overspending and underspending, but they exhibit dependence with respect to multiple unobservable parameters, like risk aversion and the risk premia on risky assets, so they are difficult to implement in real-world situations. The 4% rule, which is popular among financial advisors, is free from this complexity, but it is calibrated to the worst past economic conditions, like the stagflationary period of the early 1970s, so it is unnecessarily conservative in most scenarios and cannot be guaranteed to be feasible in the future if equity and bond returns were to fall short of historical standards.

In this paper, we argue that individuals who want to maximize the fixed (up to cost-of-living adjustment) replacement income that they can finance with their retirement savings need a retirement bond, defined as a basket of pure discount bonds with face values of $1 each and laddered redemption dates. The reciprocal of the bond price at the retirement time is the maximum withdrawal rate for the selected planning horizon, e.g. 20 or 30 years. This rate has rarely been equal to 4% and moves in line with the level of interest rates, which is evidence that a constant withdrawal rate independent of market conditions cannot meet the feasibility and the exhaustivity criteria, except by chance.

While it is tempting to replace the 4% rule with the “purchasing power rule”, in which the percentage is calculated at the retirement time from the planning horizon and the level of interest rates, this strategy is not feasible unless savings are fully invested in the retirement bond. For retirees who follow a different investment strategy, we introduce a dynamic spending policy, in which the amount withdrawn in a given year equals the amount of savings divided by the market price of the retirement bond at that time. This “maximally moderate rule”, which is the limit of Merton’s optimal consumption policy for infinite risk aversion, is suited for individuals who cannot afford the amount of replacement income that they would like and must therefore invest part of their savings in equities to have a chance of reaching their target. Through numerical illustrations, we show that the welfare loss from using that rule instead of Merton's optimal consumption and investment plan is moderate. We also show that the substitution of the aggregate bond index with the retirement bond in balanced funds and target date funds – two categories of funds that are commonly used in retirement investing – decreases the downside risk in replacement income borne by individuals who follow the maximally moderate policy.

The maximally moderate rule seems to be a promising practical alternative to utility-maximizing consumption policies and the 4% rule, but it raises several questions. First, how large is the utility loss with respect to the optimal solution? Our numerical illustration indicates that it is small in the case of constant investment opportunities, but this result should be examined in a richer setting with a stochastic opportunity set. This case is technically more challenging because no closed-form expression for indirect utility is available, but numerical solutions, including finite differences, may be a way to proceed. Second, the maximally moderate policy eliminates uncertainty over the final surplus, which is always zero, by introducing variability in withdrawals. While the use of the retirement bond, or a replicating portfolio, in a mutual fund reduces the semi-volatility of replacement income, it is insufficient to allow for fine control over downside risk. To protect individuals against severe spending cuts in bear markets, it would be useful to have investment policies that set a hard floor on maximally moderate replacement income. The third extension pertains to the investment horizon, which we keep fixed in the analysis, following the work of Merton (1969, 1971, 1973) and Bengen (1994), but lifetime is uncertain. Is there a modification to the maximally moderate rule that can be used in the uncertain horizon case? Siegel and Waring (2015) make some suggestions in this direction, but an uncertain horizon profoundly changes the nature of the risk-free asset, which is no longer the retirement bond discussed in this paper. We leave these questions for further research.

Appendix

A1 Proof of Proposition 1

Naïve Annuitization Rule

We show by induction on *t* that for any $t = 1, \dots, T$, the following equalities hold:

$$c_t = \frac{W_0}{T} R_{0,t},$$

$$W_t = \frac{T-t}{T} W_0 R_{0,t}.$$

These properties hold when $t = 1$ because

$$W_{1-} = W_0 R_{0,1},$$

so that

$$c_1 = \frac{W_{1-}}{T} = \frac{W_0}{T} R_{0,1}$$

and

$$W_1 = W_{1-} - c_1 = \left[1 - \frac{1}{T}\right] W_0 R_{0,1} = \frac{T-1}{T} W_0 R_{0,1}.$$

Now, if the properties hold for a given time $t \leq T-1$, we have

$$W_{[t+1]-} = W_t R_{t,t+1} = \frac{T-t}{T} W_0 R_{0,t+1},$$

hence

$$c_{t+1} = \frac{W_{[t+1]-}}{T-t} = \frac{W_0}{T} R_{0,t+1},$$

and

$$W_{t+1} = W_{[t+1]-} - c_{t+1} = \frac{T-t-1}{T} W_0 R_{0,t+1}.$$

Maximally Moderate Rule

We show by induction on *t* that

$$c_t = ri_{max,0} \frac{R_{0,t}}{\rho_{0,t}} CF_t,$$

and

$$W_t = ri_{max,0} \frac{R_{0,t}}{\rho_{0,t}} \beta_t.$$

Assume first that $t = 1$. Then,

$$c_1 = \frac{W_{1-}}{\beta_{1-}} CF_1 = \frac{W_0 R_{0,1}}{\beta_0} \frac{\beta_0}{\beta_{1-}} CF_1 = ri_{max,0} R_{0,1} \frac{\beta_0}{\beta_{1-}} CF_1.$$

Moreover, we have $\beta_1 = \beta_{1-} - CF_1$, so that

$$W_1 = W_{1-} - c_1 = W_{1-} - \frac{W_{1-}}{\beta_{1-}} CF_1 = W_{1-} \frac{\beta_1}{\beta_{1-}} = ri_{max,0} R_{0,1} \frac{\beta_0}{\beta_{1-}} \beta_1.$$

The total return of the retirement bond from date 0 to date 1 is

$$\rho_{0,1} = \frac{\beta_{1-}}{\beta_0},$$

which concludes the proof for $t = 1$.

Assume now that the property holds true for a given time $t \leq T - 1$. Then,

$$W_{[t+1]-} = W_t R_{t,t+1} = ri_{max,0} \frac{R_{0,t}}{\rho_{0,t}} \beta_t R_{t,t+1} = ri_{max,0} \frac{R_{0,t+1}}{\rho_{0,t}} \beta_t = ri_{max,0} \frac{R_{0,t+1}}{\rho_{0,t}} \frac{\beta_t}{\beta_{[t+1]-}} \beta_{[t+1]-}.$$

The total return on the bond from time *t* to time *t* + 1 is precisely $\beta_t / \beta_{[t+1]-}$, so, by cumulating it with the total return from 0 to *t*, we obtain

$$W_{[t+1]-} = ri_{max,0} \frac{R_{0,t+1}}{\rho_{0,t+1}} \beta_{[t+1]-}.$$

Then, by the definition of maximally moderate withdrawals,

$$c_{t+1} = \frac{W_{[t+1]-}}{\beta_{[t+1]-}} CF_{t+1} = ri_{max,0} \frac{R_{0,t+1}}{\rho_{0,t+1}} CF_{t+1}.$$

The value of post-withdrawal savings is

$$W_{t+1} = W_{[t+1]-} - c_{t+1} = ri_{max,0} \frac{R_{0,t+1}}{\rho_{0,t+1}} \beta_{t+1}.$$

So, both properties hold true at time *t* + 1, which concludes the proof.

A2 Proof of Proposition 2

Let tr denote the trace operator. By Ito's lemma, we have

$$dJ = J_t dt + J_X d\widehat{X_t} + \frac{1}{2} d < \widehat{X_t} > + J_Y' dY_t + J_{XY}' d < \widehat{X_t}, Y_t > + \frac{1}{2} tr\left[J_{YY} \Sigma_{Y,t}\right] dt,$$

$$\frac{d\widehat{X_t}}{\widehat{X_t}} = \frac{dX_t}{X_t} - \frac{d\widetilde{\beta}_t}{\widetilde{\beta}_t} + \left\|\sigma_{\beta,t}\right\|^2 dt - w_t' \sigma_t' \sigma_{\beta,t} dt.$$

$\|\sigma_{\beta,t}\|$ denotes the Euclidean norm of the vector $\sigma_{\beta,t}$.

Therefore,

$$d < \widehat{X_t}, Y_t >= \widehat{X_t}\sigma'_{Y,t}[\sigma_t w_t - \sigma_{\beta,t}]dt.$$

The conditional expectation of the change in indirect utility is

$$\begin{aligned} E_t[dJ] = J_t dt + J_X\widehat{X_t}\left[w_t'\mu_t - \mu_{\beta,t} + \|\sigma_{\beta,t}\|^2 - w_t'\sigma_t'\sigma_{\beta,t}\right]dt \\ + \frac{1}{2}J_{XX}\widehat{X_t}^2\left[w_t'\Sigma_t w_t + \|\sigma_{\beta,t}\|^2 - 2w_t'\sigma_t'\sigma_{\beta,t}\right]dt + J_Y'm_{Y,t}dt + \widehat{X_t}[\sigma_t w_t - \sigma_{\beta,t}]'\sigma_{Y,t}J_{XY}dt \\ + \frac{1}{2}tr[J_{YY}\Sigma_{Y,t}]dt. \end{aligned} \quad \text{(A1)}$$

The dynamic programming principle implies the Bellman equation

$$J(t, \widehat{X_t}, Y_t) = \max_{w_t} E_t[J(t+dt, \hat{X}_{t+dt}, Y_{t+dt}) + u(\widehat{X_t})dt],$$

which can be equivalently written as

$$\max_{w_t} E_t[dJ] = -u(\widehat{X_t})dt. \quad \text{(A2)}$$

Moreover, the optimal portfolio at time *t* is the vector w_t that maximizes the right-hand side. The first-order optimality condition reads

$$0 = J_X\widehat{X_t}[\mu_t - \sigma_t'\sigma_{\beta,t}] + J_{XX}\widehat{X_t}^2[\Sigma_t w_t^* - \sigma_t'\sigma_{\beta,t}] + \widehat{X_t}\sigma_t'\sigma_{Y,t}J_{XY}.$$

Given that $\sigma_t'\sigma_{\beta,t} = c_{\beta,t}$ and $\sigma_t'\sigma_{Y,t} = c_{Y,t}$, we obtain that

$$w_t^* = -\frac{J_X}{\widehat{X_t}J_{XX}}\Sigma_t^{-1}\mu_t + \left[1 + \frac{J_X}{\widehat{X_t}J_{XX}}\right]\Sigma_t^{-1}c_{\beta,t} - \frac{1}{\widehat{X_t}J_{XX}}\Sigma_t^{-1}c_{Y,t}J_{XY}.$$

Assume that indirect utility is separable in the relative fund value and state variables in the sense that there is a function *F* of time and state variables such that

$$J(t, \widehat{X_t}, Y_t) = \left[\frac{W_0}{\beta_0\widehat{X_0}}\right]^{1-\gamma} u(\widehat{X_t})G(t, Y_t).$$

Let $k = [W_0\widetilde{\beta_0}/\widehat{X_0}]^{1-\gamma}$ for brevity. Then,

$$J_X = k\hat{X}^{-\gamma}G = [1-\gamma]\frac{J}{\hat{X}},$$

$$J_{XX} = -k\gamma\hat{X}^{-\gamma-1}G = -\gamma[1-\gamma]\frac{J}{\widehat{X^2}},$$

$$J_{XY} = k\hat{X}^{-\gamma}G_Y = [1-\gamma]\frac{J}{\hat{X}}\frac{G_Y}{G}.$$

Hence, the optimal portfolio is given by

$$w_t^* = \frac{1}{\gamma}\Sigma_t^{-1}\mu_t + \left[1-\frac{1}{\gamma}\right]\Sigma_t^{-1}c_{\beta,t} + \frac{1}{\gamma G}\Sigma_t^{-1}c_{Y,t}G_Y.$$

A3 Partial Differential Equation for *G*

By Equations (A1) and (A2), we have

$$J_t dt + J_X\widehat{X_t}\left[w_t^{*\prime}\mu_t - \mu_{\beta,t} + \left\|\sigma_{\beta,t}\right\|^2 - w_t^{*\prime}c_{\beta,t}\right]dt + \frac{1}{2}J_{XX}\widehat{X_t}^2\left[w_t^{*\prime}\Sigma_t w_t^* + \left\|\sigma_{\beta,t}\right\|^2 - 2w_t^{*\prime}c_{\beta,t}\right]dt \tag{A3}$$

$$+ J_Y' m_{Y,t}dt + \widehat{X_t}\left[w_t^{*\prime}c_{Y,t} - \sigma_{\beta,t}'\sigma_{Y,t}\right]J_{XY}dt + \frac{1}{2}tr\left[J_{YY}\Sigma_{Y,t}\right]dt = -u\left(\widehat{X_t}\right)dt.$$

Denote with $\lambda_{MSR,t}^2 = \mu_t'\Sigma_t^{-1}\mu_t$ the squared Sharpe ratio of the maximum Sharpe ratio portfolio. From the expression of the optimal portfolio, we have

$$w_t^{*\prime}\mu_t = \frac{\lambda_{MSR,t}^2}{\gamma} + \left[1-\frac{1}{\gamma}\right]\mu_t'\Sigma_t^{-1}c_{\beta,t} + \frac{1}{\gamma G}\mu_t'\Sigma_t^{-1}c_{Y,t}G_Y,$$

$$w_t^{*\prime}c_{\beta,t} = \frac{1}{\gamma}\mu_t'\Sigma_t^{-1}c_{\beta,t} + \left[1-\frac{1}{\gamma}\right]c_{\beta,t}'\Sigma_t^{-1}c_{\beta,t} + \frac{1}{\gamma G}c_{\beta,t}'\Sigma_t^{-1}c_{Y,t}G_Y,$$

$$w_t^{*\prime}c_{Y,t} = \frac{1}{\gamma}\mu_t'\Sigma_t^{-1}c_{Y,t} + \left[1-\frac{1}{\gamma}\right]c_{\beta,t}'\Sigma_t^{-1}c_{Y,t} + \frac{1}{\gamma G}G_Y'c_{Y,t}'\Sigma_t^{-1}c_{Y,t},$$

$$w_t^{*\prime}\Sigma_t w_t^* = \frac{\lambda_{MSR,t}^2}{\gamma^2} + \left[1-\frac{1}{\gamma}\right]^2 c_\beta'\Sigma_t^{-1}c_{\beta,t} + \frac{1}{\gamma^2G^2}G_Y'c_{Y,t}'\Sigma_t^{-1}c_{Y,t}G_Y$$

$$+\frac{2}{\gamma}\left[1-\frac{1}{\gamma}\right]\mu_t'\Sigma_t^{-1}c_{\beta,t} + \frac{2}{\gamma G}\left[1-\frac{1}{\gamma}\right]G_Y'c_{Y,t}'\Sigma_t^{-1}c_{\beta,t} + \frac{2}{\gamma^2 G}\mu_t'\Sigma_t^{-1}c_{Y,t}G_Y.$$

The derivatives of *J* satisfy

$$J_X = k\hat{X}^{-\gamma}G, \qquad J_{XX} = -k\gamma\hat{X}^{-\gamma-1}G, \qquad J_{XY} = k\hat{X}^{-\gamma}G_Y,$$

$$J_t = k\frac{\hat{X}^{1-\gamma}G_t}{1-\gamma}, \qquad J_Y = k\frac{\hat{X}^{1-\gamma}}{1-\gamma}G_Y, \qquad J_{YY} = k\frac{\hat{X}^{1-\gamma}}{1-\gamma}G_{YY}.$$

Substituting these expressions into Equation (A3) and re-arranging terms, we obtain a partial differential equation for *G*:

$$0 = \frac{G_t}{1-\gamma} + G\left[\frac{\lambda_{MSR,t}^2}{2\gamma} + \left[\frac{\gamma}{2}+\frac{1}{2\gamma}-1\right]c_{\beta,t}'\Sigma_t^{-1}c_{\beta,t} + \left[1-\frac{1}{\gamma}\right]\mu_t'\Sigma_t^{-1}c_{\beta,t} + \frac{1}{2\gamma G^2}G_Y'c_{Y,t}'\Sigma_t^{-1}c_{Y,t}G_Y - \mu_{\beta,t}\right.$$

$$\left. + \left[1-\frac{\gamma}{2}\right]\left\|\sigma_{\beta,t}\right\|^2\right] + \frac{G_Y'm_Y}{1-\gamma} + \frac{1}{\gamma}\mu_t'\Sigma_t^{-1}c_{Y,t}G_Y + \left[1-\frac{1}{\gamma}\right]c_{\beta,t}'\Sigma_t^{-1}c_{Y,t}G_Y - \sigma_{\beta,t}'\sigma_{Y,t}G_Y$$

$$+\frac{1}{2[1-\gamma]}tr\left[G_{YY}\Sigma_{Y,t}\right] + \frac{1}{1-\gamma}.$$

The terminal condition is $G(T,Y) = 0$ for all *Y* because utility from future consumption is zero at the last date.

The above partial differential equation can be simplified in a “semi-complete” market setting where the retirement bond is spanned by existing securities. (The market may still be incomplete if the state variables are not perfectly hedgeable.) Then, there exists a portfolio $w_{\beta,t}$ such that $\sigma_{\beta,t} = \sigma_t w_{\beta,t}$ and $\mu_{\beta,t} = \mu_t' w_{\beta,t}$, so we have

$$c_{\beta,t} = \sigma_t' \sigma_{\beta,t} = \Sigma_t w_{\beta,t},$$

hence

$$c_{\beta,t}' \Sigma_t^{-1} c_{\beta,t} = w_{\beta,t}' \Sigma_t w_{\beta,t} = \left\| \sigma_{\beta,t} \right\|^2,$$

and

$$\mu_t' \Sigma_t^{-1} c_{\beta,t} = \mu_t' w_{\beta,t} = \mu_{\beta,t}.$$

We also have

$$c_{\beta,t}' \Sigma_t^{-1} c_{Y,t} = w_{\beta,t}' c_{Y,t} = w_{\beta,t}' \sigma_t' \sigma_{Y,t} = \sigma_{\beta,t}' \sigma_{Y,t}.$$

Thus, the equation for G simplifies to

$$\begin{aligned} 0 = \frac{G_t}{1-\gamma} + G\left[\frac{\lambda_{MSR,t}^2}{2\gamma} + \frac{1}{2\gamma}\left\|\sigma_{\beta,t}\right\|^2 - \frac{1}{\gamma}\mu_{\beta,t} + \frac{1}{2\gamma G^2} G_Y' c_{Y,t}' \Sigma_t^{-1} c_{Y,t} G_Y\right] + \frac{G_Y' m_Y}{1-\gamma} + \frac{1}{\gamma}\mu_t' \Sigma_t^{-1} c_{Y,t} G_Y \\ - \frac{1}{\gamma}\sigma_{\beta,t}' \sigma_{Y,t} G_Y + \frac{1}{2[1-\gamma]} tr\left[G_{YY} \Sigma_{Y,t}\right] + \frac{1}{1-\gamma}. \end{aligned} \tag{A4}$$

A4 Certain Equivalent Wealth Calculations

Maximally Moderate Rule

If all investment opportunities, including the short-term interest rate, are constant, the retirement bond has zero instantaneous volatility, so the partial differential equation (A4) simplifies to

$$0 = \frac{G_t}{1-\gamma} + \frac{\lambda_{MSR}^2}{2\gamma} G + \frac{1}{1-\gamma}.$$

Therefore, G is a function of time only. The terminal condition is $G(T) = 0$.

Define the auxiliary function

$$G_2(t) = G(t)\, exp\left[\frac{1-\gamma}{2\gamma}\lambda_{MSR}^2 t\right],$$

so that

$$G_{2,t} = -\,exp[a_{mm} t],$$

where $a_{mm} = -[1-\gamma]\lambda_{MSR}^2/[2\gamma]$. We have $G_2(T) = 0$, so integration gives

$$G_2(t) = \frac{exp[a_{mm}T] - exp[a_{mm}t]}{a_{mm}},$$

hence

$$G(t) = \frac{1 - exp[-a_{mm}[T - t]]}{a_{mm}}.$$

Certain Equivalent Wealth Under Naïve Annuitization Rule

When investment opportunities are constant, the optimal portfolio under Merton's consumption rule, the maximally moderate rule or the naïve annuitization rule is

$$w = \frac{1}{\gamma}\Sigma^{-1}\mu,$$

so that the fund value evolves as

$$\frac{dX_t}{X_t} = [r + w'\mu]dt + w'\sigma' dz_t = \left[r + \frac{\lambda_{MSR}^2}{\gamma}\right]dt + \frac{1}{\gamma}\lambda' dz_t,$$

where $\lambda_{MSR}^2 = \mu'\Sigma^{-1}\mu$ is the squared maximum Sharpe ratio of the tangency portfolio, and $\lambda = \sigma\Sigma^{-1}\mu$ is the "price of risk vector". It follows from Ito's lemma that

$$E[ln\, X_t] = ln\, X_0 + \left[r + \frac{\lambda_{MSR}^2[2\gamma - 1]}{2\gamma}\right]t$$

and variance

$$V[ln\, X_t] = \frac{\lambda_{MSR}^2}{\gamma^2}t.$$

Under the naïve annuitization rule, withdrawals are given by

$$c_{na,t} = \frac{W_0}{T}\frac{X_t}{X_0},$$

so they are log-normally distributed, like X_t, and we have

$$E[u(c_{na,t})] = u\left(\frac{W_0}{T}\right) exp\left[[1-\gamma]E\left[ln\frac{X_t}{X_0}\right] + \frac{[1-\gamma]^2}{2}V\left[ln\frac{X_t}{X_0}\right]\right] = u\left(\frac{W_0}{T}\right) exp\left[[1-\gamma]\left[r + \frac{\lambda_{MSR}^2}{2\gamma}\right]t\right].$$

Define

$$a_{na} = -[1-\gamma]\left[r + \frac{\lambda_{MSR}^2}{2\gamma}\right].$$

Then,

$$E\left[\int_0^T u(c_{na,t})dt\right] = u\left(\frac{W_0}{T}\right)\int_0^T exp[-a_{na}s]\,ds = u\left(\frac{W_0}{T}\right)\frac{1-exp[-a_{na}T]}{a_{na}}.$$

The certain equivalent wealth is defined as

$$u(W_{eq,na}) = E\left[\int_0^T u(c_{na,t})dt\right],$$

hence

$$W_{eq,na} = \frac{W_0}{T}\left[\frac{1-exp[-a_{na}T]}{a_{na}}\right]^{1-\gamma}.$$

Certain Equivalent Wealth Under Maximally Moderate Rule

Under the maximally moderate rule, consumption is

$$c_{mm,t} = \frac{W_0}{\beta_0}\frac{X_t\widetilde{\beta_0}}{X_0\widetilde{\beta_t}}.$$

With a constant short-term rate, the total return on the retirement bond equals cash return, hence

$$c_{mm,t} = \frac{W_0}{\beta_0}\frac{X_t}{X_0}e^{-rt}.$$

Using the log-normality of X_t, we obtain

$$E[u(c_{mm,t})] = u\left(\frac{W_0}{\beta_0}\right)exp[-a_{mm}t],$$

with

$$a_{mm} = -[1-\gamma]\frac{\lambda_{MSR}^2}{2\gamma}.$$

Therefore,

$$E\left[\int_0^T u(c_{mm,t})dt\right] = u\left(\frac{W_0}{\beta_0}\right)\frac{1-exp[-a_{mm}T]}{a_{mm}},$$

and the expression for $W_{eq,mm}$ follows.

Certain Equivalent Wealth Under Merton's Rule

Finally, the expression for the certain equivalent wealth under Merton's rule can be obtained by solving the partial differential equation (7) of Liu (2007) when there are no state variables. In Liu (2007), indirect utility is $u(W_0)f(0)^\gamma$, where *f* solves

$$f_t - a_{me}f = -1,$$

and

$$a_{me} = -\left[\frac{1-\gamma}{2\gamma^2} + \frac{1-\gamma}{2\gamma} r\right].$$

Integrating the differential equation with the terminal condition $f(0) = 0$, we obtain

$$I(0, W_0) = u(W_0)\left[\frac{1 - exp[-a_{me}T]}{a_{me}}\right]^{\gamma},$$

hence

$$W_{eq,me} = W_0\left[\frac{1 - exp[-a_{me}T]}{a_{me}}\right]^{\frac{\gamma}{1-\gamma}}.$$

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