Mitigating the Newsvendor Problem through Call Option Contracts in Decentralized Supply Chains

Pooya Hedayatinia[a], Guillaume Massonnet[b], Jean-Laurent Viviani[a], David Lemoine[b]

[a] *Univ Rennes, CNRS, CREM - UMR6211, Rennes F-35000, France (e-mail: {pooya.hedayatinia, jean-laurent.viviani}@univ-rennes1.fr)*
[b] *IMT Atlantique, LS2N, 4 rue Alfred Kastler, 44300 Nantes, France (e-mail: {david.lemoine, guillaume.massonnet}@imt-atlantique.fr)*

Abstract

The management of uncertain demand is a fundamental aspect of operations management, particularly in decentralized supply chain structures. This poses the newsvendor problem, which represents a significant challenge for retailers facing uncertain future demand, resulting in decreased performance throughout the entire supply chain. This challenge exists in financial markets and one of the ways to deal with it is through option contracts. In this study, we examine the impact of incorporating a call option contract in addition to the traditional wholesale price contract. Our analysis demonstrates that the inclusion of call option contracts can improve supply chain performance and mitigate the effects of double marginalization. These contracts offer a means for the supplier to strategically assign risk among supply chain partners, ultimately leading to improved overall performance. Furthermore, we demonstrate that, for the supplier to maximize expected profits, they must assume the majority of demand risk.

Keywords: Option contracts, supply chain, Newsvendor, risk management

1. Introduction

In today's economy, various industries are grappling with the challenges posed by demand uncertainty, which can negatively impact supply chain performance. The newsvendor model, a widely-used framework for addressing uncertainty in demand, deals with a retailer who must decide on an inventory level for a future selling season with uncertain demand. The challenge for the retailer is to determine an optimal order quantity while accounting for the costs associated with over-stocking or under-stocking. When demand is deterministic, the newsvendor can easily decide on the number of units needed for the selling season. However, introducing uncertainty in demand can decrease performance by reducing or increasing inventory levels and final sales. In fact, when demand is uncertain, the optimal inventory level depends on each unit's profit margin and its opportunity cost. In a decentralized supply chain, it is only the retailer who faces demand uncertainty. This leads to Pareto sub-optimality, as the retailer orders less than a centralized supply chain optimal production, resulting in lower supplier sales and fill rates for customers. To address this problem, the supplier may choose to share the risk of demand uncertainty with the retailer. In finance, it is well-known that derivatives such as contracts and options can be used to transfer risk between investors. This paper aims to investigate the impact of introducing an option contract between the supplier and retailer on the decisions, profits, and overall efficiency of the supply chain.

In capital markets, option contracts are commonly used as a derivative tool to hedge against future price and demand uncertainty (Hull 2003). These contracts involve a seller, known as the option writer, and a buyer, who pays for the contract and has the option to exercise it as needed. Similarly, in operations management (OM), option contracts can be used to manage inventory risk, with the supplier and retailer acting as the writer and buyer, respectively. In this study, we focus on the call option contract, which involves a prepayment and later delivery of the product along with an additional payment. These option contracts are similar to another type of contract commonly used in OM, known as capacity reservation contracts. However, a key difference between the two is that in

*This work was partially supported by the French National Research Agency and this is the project ANR-17-CE10-0001-01 called *FILEAS FOG*.

capacity reservation contracts, the retailer can finalize his order at a later stage after initial ordering (Gomez-Padilla and Mishina 2013).
In this analysis, we examine a supply chain consisting of a supplier acting as the Stackelberg leader and a newsvendor-style retailer as the Stackelberg follower. We investigate a scenario where the supplier offers both a classical wholesale price contract and a call option contract to the retailer. The retailer has the ability to use one or both contracts to procure inventory for the selling season. Through our analysis, we demonstrate how the retailer can make optimal decisions regarding inventory quantities and how the supplier can design contracts to maximize her expected profits. We show that the retailer can always find a unique optimal quantity to maximize his expected profit, while the profit-maximizing supplier uses the option contracts to absorb demand uncertainty, ultimately maximizing her expected profit function.

2. Literature review

In recent years, the increasing development of industries and globalization has led to a greater prevalence of decentralized supply chains, where suppliers sell their products through independent retailers to reach the final customers. However, as identified by Spengler 1950, this decentralization can lead to a reduction in overall system performance due to the double marginalization effect (Li et al. 2013). This occurs as each supply chain partner maximizes their own expected profit, leading to a retailer's optimal order quantity that is lower than what would be observed in a centralized supply chain (Li et al. 2013). Furthermore, in a decentralized structure, the retailer is the only entity facing the final demand, which may be uncertain, and they may lack the flexibility to effectively manage this uncertainty through traditional supply chain contracts (Van Mieghem 1998, Birge 2000). To address these challenges, option contracts have been proposed as a means of sharing the risk of uncertain demand with the retailer and providing greater flexibility in the ordering process (Eriksson 2019, Feng et al. 2014, Basu et al. 2018). Wang and Chen 2015 have demonstrated that option contracts are particularly preferable for newsvendor-type retailers operating in the context of high demand uncertainty. In this paper, we aim to investigate the potential benefits of implementing option contracts in a decentralized supply chain, with a specific focus on the mitigation of the double marginalization effect and the management of uncertain demand.
The option contracts are well-known in finance and vastly used to dealing with price and demand uncertainties Hull 2003. Classically, there are two kinds of option contracts in finance, call option contracts which are similar to capacity reservation contracts Martínez-de Albéniz and Simchi-Levi 2009 and the put option contracts which is the general form of the buy-back contracts in OM Chen and Parlar 2007, Basu et al. 2018. There exists another kind of option contract in OM which is a bidirectional option contract. This contract can convert to a put or call option at the time of exercising it Wang and Tsao 2006, Wang and Chen 2021.
The option contracts are already existed through the industries, for example the agriculture industry has been using option contracts through the Chicago Mercantile Exchange to hedge against the weather fluctuations Xue et al. 2015. Hewlett Packard (HP) has been using these contracts since 2000 to deal with their inventory risk Nagali et al. 2008. Adapting the option contracts in OM is getting more attention these days Wang et al. 2020 and they are being analyzed in different ways. On this path, Liang et al. 2012 implied option contracts into relief material management as a risk management tool. There are several studies on option pricing in OM, Burnetas and Ritchken 2005 analyzed it under a demand curve with a downward slope. Jörnsten et al. 2013 investigated option contract with mixed contract with option under a discrete demand, they showed mixed contract is superior to only option contract when the supplier is risk averse. Zhao et al. 2018 studied option pricing with the existence of spot market and information updating, they introduced a new concept to evaluate the expected benefit for each unit of the option in different market situations. As the potion contracts are a risk-hedging tool for the retailer, Chen and Parlar 2007, Chen et al. 2014 and Basu et al. 2018 used a risk- averse retailer in their model. In this content, Zhuo et al. 2018 goes to details of a retailer under a call option contract with a different risk attitude, they showed that optimal decisions with option contracts are dependent on the supply chain partner's risk attitude. Wang and Chen 2017, Zhuo et al. 2018 and Chen et al. 2014 analyzed the conditions to reach supply chain coordination, which means that the members of the supply chain optimized decisions to maximize the whole system's expected profit. Other features such as fairness concerns Sharma et al. 2019 or service requirements Chen and Shen 2012 assessed with the presence of the option contracts. The same structure of this paper, Wang and Chen 2017 and Feng et al. 2014 considered a portfolio of

contracts including a call option and a wholesale price contract, but here we analytically show how the supplier can reach a better profit and the structure that the risk is moving from the retailer to the supplier.
It is shown that the supply chain can be better off with option contracts for both the supplier and the retailer Burnetas and Ritchken 2005. Yang et al. 2017 illustrates the inventory risk reduction implied by these contracts. It is usual to consider the supplier as Stackelberg leader in the supply chain, however Liu et al. 2020 analyze them with a retailer leader as well. Our study considers the supplier as the leader and he is a profit maximizer without risk attitude consideration. We analyze this structure analytically under a stochastic demand and shows the details of the retailer and supplier. The contribution of this study is showing how the supplier can reach higher expected profit and illustrating a crucial point which is maximizing the expected profit of the supplier is associated with maximizing absorption of demand uncertainty.

3. Model Description

In this study, we consider a simple supply chain comprising a supplier (Stackelberg leader) and a newsvendor-type retailer, both of whom are profit maximizers. The retailer is required to procure inventory for a future stochastic demand, D, at time 1. The supplier offers two types of contracts to the retailer: a wholesale price contract, where the retailer pays w for each ordered unit at time 0 and receives them at time 1, and a call option contract, where the retailer pays the option price o for each ordered unit at time 0 and, at time 1, pays the exercise price e for each unit willing to receive. The supplier designs these contracts such that $o + e > w$ and $o < w$, thereby providing the retailer with an incentive to utilize both contracts. At the end of the time horizon, the retailer incurs a loss of w for each unsold unit under the wholesale price contract or o for each unsold unit under the option contract. As $o < w$, the loss for each unsold unit under the option contract is less than that under the wholesale contract. Therefore, the retailer satisfies demand through the wholesale contract and only after exhausting all wholesale quantities, starts utilizing the option contract. The stochastic demand is defined by the probability density function (pdf) of D, denoted by $\phi(\cdot)$, and its cumulative distribution function (cdf), denoted by $\Phi(\cdot)$. We assume that D has an increasing failure rate (IFR), i.e. the function $h : x \mapsto \phi(x)/\bar{\Phi}(x)$ is increasing with respect to x.
At time 0, first the supplier needs to choose the parameters of his contracts, wholesale price w for the wholesale contract, option price o and exercise e exercise price for the option contract. Then she offers them to the retailer who has to decide how many needs to order and the level between ordering from the wholesale contract and option contract. At the selling season time 1, the retailer decides to exercise how many of the option contracts by realizing the demand. The retailer first serves the demand from his wholesale contract as it has a higher profit margin and higher costs for the unsold products, then he refers to the option contracts. The retailer collects the price of p for each unit of satisfied demand, regardless of the used contract to procure that unit. As a summary, the supplier has to decide about w, o and e to maximize her expected profit and she cannot change the cost of production c. The retailer maximizes his expected profit function through deciding on wholesale quantity q_w and the option quantity q_c, while the selling price p is fixed by the market. Symbols with their descriptions are summarized in Table 1.

Symbol	**Description**
p	Retailer's selling price of each product.
w	Wholesale price in wholesale price contract.
o	Option price of the option contract.
e	Exercise price of the option contract.
c	Cost of production of the supplier.
q_w	Retailer's wholesale price order quantity.
q_c	Retailer's option order quantity.
$\mathbf{q} = (q_c, q_w)$	Decision vector of the retailer.
D	Customers stochastic demand of the product at time T.
$\phi(.), \Phi(.), \bar{\Phi}(.)$	pdf, cdf and complementary cdf of D, respectively.

Table 1: Notations and symbols

So the retailer has to decide jointly the option quantity q_c and the wholesale price quantity q_w, where $\mathbf{q}$ represents the decision his vector at time 0. Once he observes the realized demand d at time 1, he begins by serving it using the units available from the wholesale price decision q_w, then exercises his option to purchase $\min\{q_c, \max\{d - q_w, 0\}\}$ additional units at the exercise price e if it is necessary. We assume that $0 < o < w < o + e < p$ to avoid trivial cases and no contract Pareto-dominates the other one. Actually, $o < w$ has to be satisfied that the retailer can use the option as well, otherwise he orders only from the wholesale contract. The assumption of $w < o+e < p$ ensures that the retailer uses both contracts and each sold unit from any of the contracts, generates a profit for him. The last point, we know in finance that we need to always consider the time value of the money. In this model to sum the payments of time 0 and 1, we can introduce a risk-free rate as the time value of money. However, we do not have an interest rate else the risk-free rate, thus as a matter of simplicity, we consider the risk-free rate equal to zero. The sequence of the retailer and supplier are briefly explained in Table 1.
We start the analysis with the retailer because the supplier is the leader in our model and his optimal decision is dependent on the retailer's reaction to the contracts. So we need to first understand how the retailer reacts to any offered contract and then we can analyze the supplier.

3.1. The retailer decision

First we have to write the retailer's expected profit function which includes his cost in total and revenue at time 1. It is necessary to consider that the retailer first used all his quantity from the wholesale contract to serve the demand, then he starts to exercise his option contracts. This is because he has a higher profit margin and cost of unsold product with the wholesale contract. So we can express the expected profit as below:

$$\begin{aligned} \Pi_r(\mathbf{q}) &= \mathbb{E}\left[p\min\{q_w + q_c, \mathrm{D}\} - e\min\{(\mathrm{D} - q_w)^+, q_c\}\right] - wq_w - oq_c \\ &= p\int_0^{q_w+q_c} \bar{\Phi}(u)du - e\int_{q_w}^{q_w+q_c} \bar{\Phi}(u)du - wq_w - oq_c \end{aligned} \tag{1}$$

In equation (1), the first term gives the maximum possible level of sales, the second term the extra cost associated with the exercise of the option contract, the two last terms the total cost paid at the beginning of the period. The retailer needs to find a pair of quantities q_w and q_c to maximize his expected profit given in equation (1). The proposition below illustrates the decision of the retailer:

Proposition 1. *The retailer can decide jointly unique quantities of q_w and q_c to maximize his expected profit function. The optimal option and wholesale price order quantity is [Proof in Appendix]:*

$$q_w^* + q_c^* = \bar{\Phi}^{-1}\left(\frac{o}{p-e}\right) \ , \ \ q_w^* = \bar{\Phi}^{-1}\left(\frac{w-o}{e}\right) \tag{2}$$

We can see the retailer can always choose his quantities as long as our assumptions of $w < o+e < p$ is satisfied. The quantities in the equation (2) always maximize the expected profit of the retailer and it is the unique maximum of it. As this answer is a unique maximum and the only wholesale price contract is a special case of our model, it generates a higher expected profit for the retailer. So with the same wholesale price, the existence of a call option contract improves the expected profit of the retailer. In compared to the classical wholesale price contract solution $\bar{\Phi}^{-1}(\frac{w}{p})$, here the retailer orders higher total quantity and reaches a better expected profit.

1	At time 0, the supplier decides about her contract parameters and offers it to the retailer.
2	At time 0, the retailer chooses his optimal quantities, then he submits his order and pays $oq_c + wq_w$ to the supplier.
3	At time 1, the supplier delivers all products ordered from the wholesale contract.
4	At time 1, the retailer realizes the demand and first satisfies demand from the wholesale contract, then he starts exercising option to satisfy demand and pays $e\min\{(\mathrm{D} - q_w)^+, q_c\}$.
5	At time 1, the supplier receives $e\min\{(\mathrm{D} - q_w)^+, q_c\}$ and deliver $\min\{(\mathrm{D} - q_w)^+, q_c\}$. through exercising option contract.

Table 2: Time line of the events

3.2. The supplier decision

On the other side of the supply chain, the supplier has to decide about his decision variables ,i.e. the contract parameters w, o and e. First step, we can write the expected profit of the supplier as below:

$$\begin{aligned}\Pi_s(w,o,e) =& oq_c + wq_w + \mathbb{E}\left[e\min(q_c, \mathrm{D} - q_w)^+\right] - c(q_w + q_c) \\ =& oq_c + wq_w + e\int_{q_w}^{q_w+q_c} \bar{\Phi}(u)du - c(q_w + q_c)\end{aligned} \tag{3}$$

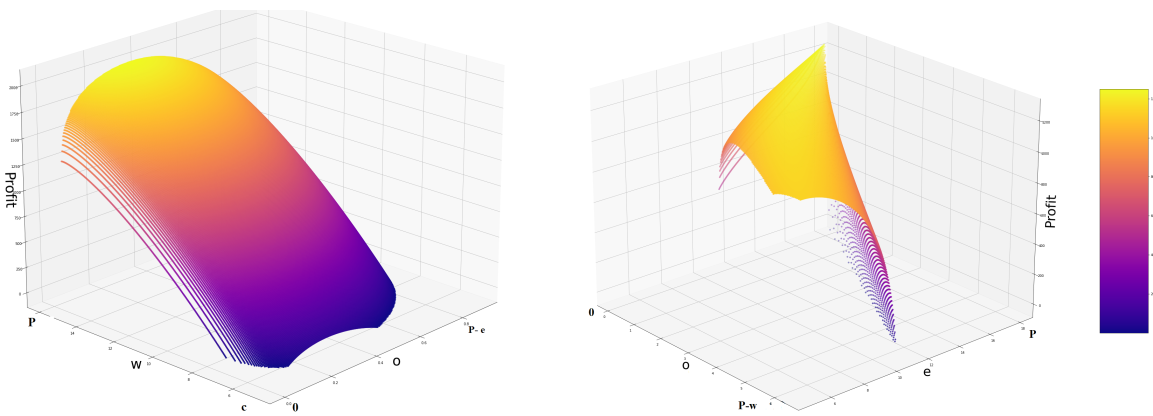
Amounts paid by the retailer are received by the supplier, explaining the three first terms of equation (3), the last term represents the total production cost of the supplier. The suppler can maximize his expected profit through three variables that she controls and for any combination of these three variables, the retailer has an unique reaction through his quantity decisions. So here, first we analyze the supplier behavior for the scenario when w is fixed and she can only manipulate option contract parameters that could be the case if the supplier is on a competitive market. The proposition below explains the supplier's behavior when she needs to decide about only o and e:

Proposition 2. *With the assumption of IFR demand distribution, the supplier's expected profit function is concave to e for a given o and vice versa, but it is not jointly concave[Proof in Appendix].*

We can see that the supplier can not find both o and e at the same time to maximize his expected profit. The next step is that which combination of o and e can generate a better expected profit. The following proposition paves the road in this aspect:

Proposition 3. *The supplier can reach a better expected profit by maximizing e, i.e. $e \to p$ and optimizes his expected profit with regarding to o[Proof in Appendix].*

The proposition 3 shows that the supplier chooses a e as high as possible and we know that $o + e$ cannot be greater than p. So as e is close to p, the optimal o will be close to 0. This means the retailer can reserve a quantity production with a very low payment, but the profit of selling this unit mostly belong to the supplier and the retailer only benefits from a very low profit margin. In another viewpoint, we can say that the supplier bears the most of demand's uncertainty risk through the option contract and the retailer bear a very small part of this risk. Thus, the most part of profit of each selling unit via call option contract belongs to the supplier.



(a) The supplier expected profit surface in o and w. (b) The supplier expected profit surface in o and e.

Figure 1: The supplier expected profit surface.

To reach comprehensive decision for the supplier, now we consider the supplier can change w as well. The next proposition reveals the supplier's decision for the wholesale price contract:

Proposition 4. *The expected profit function of the supplier is an increasing function of w with a given o and e. So the supplier chooses w as high as possible.*

As a summary, the supplier's decision is that she choose a w and e as high as possible, she may have restriction on these decision based on market competition or his capital structure or etc. Then

she can maximize his expected profit with finding an optimal o. We can see effect of these decisions on the figure 1 which illustrates the 3D surface of supplier's expected profit.
We can see the supplier is taking the most part of total profit margin $p - c$. This can be explained that the supplier is absorbing the most of demand risk through the option contract, so she can take the most of the profit. On the other hand, the retailer who bears a very small part of demand risk, only a small part of profit would reach him. We have to mention that the supplier is the leader in our model, both partner of the supply chain are profit maximizer but the supplier has the upper hand and can absorb most of risk and the profit associated with it. Thus, the option contract brings a possibility to the supply chain to pool the demand risk and the leader of this model decides how risk divides through the supply chain and higher risk associated to higher profit. The last point is when $e = p$ and $o = 0$ is resemblance of centralized supply chain and the supplier has a direct access to the final demand.

4. Conclusion

The uncertainty of future demand poses a significant challenge for inventory decisions, especially due to the demand uncertainty, the retailer's optimal inventory level depends on profit margins. In a decentralized supply chain, the total profit margin is divided among the supplier and retailer, which can lead to the double marginalization effect and a reduction in overall performance. The retailer is the only supply chain partner that is exposed to uncertain future demand. As a result, retailers often reduce their inventory levels in the face of higher uncertainty, which negatively impacts the expected profit for the entire supply chain. Under a traditional wholesale price contract, all demand uncertainty is borne by the retailer and does not provide flexibility in inventory levels during the selling season.
In this study, we demonstrate that the existence of a call option contract in addition to a classical wholesale price contract can improve supply chain performance and mitigate the uncertainty and double marginalization effect. The retailer is better off with the inclusion of a call option contract at the same wholesale price, as they can always find a combination of quantities to maximize their expected profit. The supplier, as the Stackelberg leader, has the ability to absorb demand risk through the use of a call option contract. We show that the profit-maximizing supplier chooses to absorb all demand risk and profit margin to maximize their expected profit. Regardless of whether the wholesale price is controlled by the supplier or fixed by the market, the supplier chooses an option contract with the highest possible exercise price in order to maximize their expected profit through the option price. In summary, by offering a call option contract in addition to a traditional wholesale price contract, the supplier can absorb a portion of the retailer's profit margin without reducing the order quantity. The retailer, in turn, benefits from this contract by using it as a tool to hedge against risk and increase their final profit.

Further research

Option contracts can be used in operation management to manage demand uncertainty and fluctuations in price. These uncertainties can affect the profit of supply chain partners and may lead to financial limitations or bankruptcy. This can be a important point to explore, financial limitation leads to debt financing and leverage the supply chain partners. Thus the effect of these contracts in risk management in these situation can be a very significant point to explore. The advantages of these contracts can be even greater in multi-period settings where uncertainty increases over time. Additionally, option contracts can also be used to address information asymmetry issues within a supply chain by providing a two-part payment structure.

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Appendix A. Proofs

Appendix A.1. Proof of Proposition 1

The expected profit function in 1 is:

$$\Pi(\mathbf{q}) = p\int_0^{q_w+q_c} \bar{\Phi}(u)du - e\int_{q_w}^{q_w+q_c} \bar{\Phi}(u)du - wq_w - oq_c \tag{A.1}$$

First we calculate the first derivative of the retailer's expected profit function in 1 to q_w and q_c:

$$\begin{aligned}
\frac{\partial \Pi(\mathbf{q})}{\partial q_c} &= (p-e)\bar{\Phi}(q_w+q_c) - o = 0\\
q_w^* + q_c^* &= \bar{\Phi}^{-1}(\frac{o}{p-e})\\
\frac{\partial \Pi_r(\mathbf{q})}{\partial q_w} &= (p-e)\bar{\Phi}(q_w+q_c) + e\bar{\Phi}q_w - w = 0\\
q_w^* &= \bar{\Phi}^{-1}(\frac{(w-o)}{e})
\end{aligned} \tag{A.2}$$

Then we analyze the concavity by analyzing the second derivative:

$$\begin{aligned}
\frac{\partial^2 \Pi(\mathbf{q})}{\partial {q_c}^2} &= -\,(p-e)\phi(q_w+q_c) < 0\ ,\ \frac{\partial^2 \Pi(\mathbf{q})}{\partial {q_w}^2} = -(p-e)\phi(q_w+q_c) - e\phi(q_w) < 0\\
\frac{\partial^2 \Pi(\mathbf{q})}{\partial q_c \partial q_w} &= -\,(p-e)\phi(q_w+q_c)
\end{aligned}$$

We can see the expected profit is concave in q_w and q_c, so the equations in (A.2) are showing the maximum of it. At the end by calculating the determinant of Hessian matrix which is always positive and we can see that the expected cash flow of the retailer is jointly concave to q_c and q_w:

$$|H| = e(p-e)\phi(q_w)\phi(q_w+q_c) > 0$$

So the retailer always can find the unique quantities to maximizes her expected profit.

Appendix A.2. Proof of proposition 2

We can rewrite the expected profit function of the supplier as a function of $Q^* = q_w^* + q_c^*$ and q_w^*:

$$\Pi_s(w,o,e) = oQ^* + (w-o)q_w^* - cQ^* + e\int_{q_w^*}^{Q^*} \bar{\Phi}(u)du$$

The supplier can calculate her total optimal quantity,as her expected profit is concave to Q^*:

$$\begin{aligned}
\frac{\partial^2 \Pi_s}{\partial {Q^*}^2} &= -e\phi(Q^*) < 0\\
Q_s^* &= \bar{\Phi}^{-1}(\frac{c-o}{e})
\end{aligned}$$

We can state that the optimal total quantity of the supplier is always greater than the optimal total quantity that the retailer ordered:

$$\begin{aligned}
Q_s^* = \Phi^{-1}(1-\frac{c-o}{e}) &> q_w^* + q_c^* = \Phi^{-1}(1-\frac{o}{p-e})\\
&\Rightarrow po - cp + ce > 0
\end{aligned}$$

Here, we calculate the second derivatives of the supplier's expected profit to o:

$$\begin{aligned}
\frac{\partial \Pi_s(w,o,e)}{\partial o} =& Q^* - q_w^* + \frac{po-cp+ce}{p-e}\frac{\partial Q^*}{\partial o}\\
=& Q^* - q_w^* - \frac{po-cp+ce}{(p-e)^2\phi(Q^*)}\\
\frac{\partial^2 \Pi_s(w,o,e)}{\partial o^2} =& (\frac{2p-e}{p-e})\frac{\partial Q^*}{\partial o} - \frac{\partial q_w^*}{\partial o} + (\frac{po-cp+ce}{p-e})\frac{\partial^2 Q^*}{\partial o^2}\\
=& \frac{-1}{(p-e)^2\phi^3(Q^*)}\left[(2p-e)\phi^2(Q^*) + (\frac{po-cp+ce}{o})\bar{\Phi}(Q^*)\frac{\partial \phi(Q^*)}{\partial Q^*}\right] - \frac{1}{e\phi(q_w^*)}
\end{aligned}$$

We can easily show that $(2p-e) \geq \frac{po-cp+ce}{o}$ and with IFR distribution $(\phi^2(x)+\frac{\partial\phi(x)}{\partial x}\bar{\Phi}(x)>0)$, the second derivative of the expected profit of the supplier to o is negative. So the expected profit of the supplier is concave to o.
On the next step, we calculate the second derivatives of the supplier's expected profit to e:

$$\begin{aligned}
\frac{\partial \Pi_s(w,o,e)}{\partial e} =& \frac{po-cp+ce}{p-e}\frac{\partial Q^*}{\partial e} + \int_{q_w^*}^{Q^*} \bar{\Phi}(u)du \\
=& \frac{po-cp+ce}{(p-e)^3}\frac{-o}{\phi(Q^*)} + \int_{q_w^*}^{Q^*} \bar{\Phi}(u)du \\
\frac{\partial^2 \Pi_s(w,o,e)}{\partial e^2} =& \left[\frac{o}{(p-e)^2}\right]\frac{\partial Q^*}{\partial e} + \frac{po-cp+ce}{p-e}\frac{\partial^2 Q^*}{\partial e^2} - \frac{w-o}{e}\frac{\partial q_w^*}{\partial e} \\
=& -\frac{-o(c+o)}{(p-e)^3\phi(Q^*)} - \frac{(po-cp+ce)o}{(p-e)^4\phi^3(Q^*)}\left[3\phi^2(Q^*) + \frac{\partial\phi(Q^*)}{\partial Q^*}\bar{\Phi}(Q^*)\right] - \frac{(w-o)^2}{e^3\phi(q_w^*)}
\end{aligned}$$

We can see that with IFR distribution $(\phi^2(x)+\frac{\partial\phi(x)}{\partial x}\bar{\Phi}(x)>0)$, the second derivative of the expected profit of the supplier to e is negative and the expected profit of the supplier is concave to e.

Now, we can calculate the determinant of the Hessian matrix, in order to analyze the joint concavity of the expected profit function:

$$\begin{aligned}
|H| =& \left(\frac{\partial^2 \Pi_s(w,o,e)}{\partial o^2}\right)\left(\frac{\partial^2 \Pi_s(w,o,e)}{\partial e^2}\right) - \left(\frac{\partial^2 \Pi_s(w,o,e)}{\partial e \partial o}\right)^2 \\
=& -\left(\frac{po-cp+ce}{(p-e)^2}\right)^2\left(\frac{\partial Q^*}{\partial o}\right)^2 \leq 0
\end{aligned}$$

As we can see, the determinant of the Hessian matrix is always negative and for the point that $\frac{\partial \Pi_s(w,o,e)}{\partial o}=0$ and $\frac{\partial \Pi_s(w,o,e)}{\partial e}=0$, it can be a saddle point, not a local maximum. Thus, we can state that the expected profit of the supplier is not jointly concave to e and o.

Appendix A.3. Proof of proposition 3

Here we can have the first derivative to o:

$$\begin{aligned}
\frac{\partial \Pi_s(w,o,e)}{\partial o} =& Q^* - q_w^* - \frac{po-cp+ce}{(p-e)^2\phi(Q^*)} = 0 \\
\Rightarrow& Q^* - q_w^* = \frac{po-cp+ce}{(p-e)^2\phi(Q^*)}
\end{aligned} \tag{A.3}$$

The first derivative to e:

$$\begin{aligned}
\frac{\partial \Pi_s(w,o,e)}{\partial e} =& \frac{po-cp+ce}{(p-e)^3}\frac{-o}{\phi(Q^*)} + \int_{q_w^*}^{Q^*} \bar{\Phi}(u)du \\
[\textbf{if } \frac{\partial \Pi_s(w,o,e)}{\partial o} = 0] \Rightarrow& = -\bar{\Phi}(Q^*)(Q^*-q_w^*) + \int_{q_w^*}^{Q^*} \bar{\Phi}(u)du > 0
\end{aligned} \tag{A.4}$$

As easily we can show that $\int_{q_w^*}^{Q^*} \bar{\Phi}(u)du > \bar{\Phi}(Q^*)(Q^*-q_w^*)$, the first derivative to e is always positive when $\frac{\partial \Pi_s(w,o,e)}{\partial o}=0$.

Appendix A.4. Proof of proposition 4

We can calculate the first derivative of Π_s to w with a given o and e:

$$\frac{\partial \Pi_s(w)}{\partial w} = q_w \tag{A.5}$$

We can see that the expected profit of the supplier is always an increasing function in w.