Risk management of margin based portfolio strategies for dynamic portfolio insurance with minimum market exposure

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Abstract

We extend the standard Constant Proportion Portfolio Insurance (CPPI) by introducing simultaneously margin based dynamic strategies and constraints on minimum market exposure. This leads us to introduce specific conditional floors, allowing the portfolio of not being monetized (to avoid the cash-lock risk) while ensuring better participation in potential market increases. To control the risk of such strategies, we introduce risk measures based both on quantile conditions. Our empirical analysis is mainly conducted on S&P 500 and Euro Stoxx 50, by using Monte-Carlo experiments based on circular block boostrap method. This allows us to analyze the impact of the different parameters that define our CPPI strategies (i.e. CPPI multiple, successive margins, level of the minimum market exposure). We estimate and compare the cumulative distribution functions of the portfolio returns corresponding to the various insurance strategies that we investigate. We provide also their first four moments and measure their respective performances using both the Sharpe and the Omega ratios. Our results highlight the benefits of introducing time-varying floors associated to a decreasing sequence of margins while keeping the market exposure above a minimum level.

Key words: Portfolio insurance; CPPI strategy; time vaying floor; margin based strategy; market exposure.

JEL classification: C 22, C 61, G 11.

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1 Introduction

The recent events[1] and the history of the financial markets[2] point out both the plausibility and the severity of the potential losses that an investor can experience. For risk adverse investors, such as insurers subject to regulatory constraints or pension funds with defined contributions, the control of the downside risk plays a key role in their investment process. However, downside risk control requires to address both statistical and practical concerns. First, estimating the downside risk represents an important challenge due to the structure of asset prices ([1]). For instance, the non-homogeneous behavior of asset returns implies that the risk is time-varying and need to be dynamically assessed. Second, from a practical point of view, downside risk control is not as simple as withdrawing the capital from a position at risk. Indeed, due to the market structure, a portfolio manager cannot reallocate his entire portfolio at once due to liquidity issue or without suffering from an execution risk. In addition, a significant reduction in exposure at the source of the risk significantly limits the potential benefit of a future market recovery..

In this framework, the concept of portfolio insurance has been developed to limit the portfolio downside risk while maintaining a certain upside participation. There are two main approaches to portfolio insurance: (i) the option based strategies usually known as the *option based portfolio insurance* (OBPI) and the (ii) floor-based strategies covering the well known, *constant proportion portfolio insurance* (CPPI) and *time invariant portfolio insurance* (TIPP) strategies.

Option-based strategies use option instruments to target a desired payoff profile at a given time. First introduced by Leland and Rubinstein ([2]), with the use of European put options to guarantee a minimal portfolio value in the future, these strategies have evolved considerably over time, mainly with the use of hedging strategies to mitigate the insurance cost or to benefit from more exotic option payoff profile. For instance, Föllmer et al. ([3], [4]) introduce a quantile hedging framework for investor facing budget constraints but requiring to achieve a specific goal. In a similar manner, Strassberger ([5]) implements a dynamic risk budgeting strategy based on the replication of a synthetic put to hedge the value at risk and the expected shortfall. Alternatively, Carr et al. ([6]) focus on hedging the maximum drawdown using double barrier options. Additionally, since double barrier options are not liquid, they provide alternative hedging strategies based on more vanilla options. Other different approaches rely on optimization procedures to select the structure of the option strategies. Capinski ([7]) proposes to find the optimal allocation of put options that minimize the conditional value at risk of a portfolio subject to a cost constraints.

Although they offer a wide range of solutions, option based strategies are not always easily implementable. On one hand, options are not necessarily liquid instruments and even the most liquid option markets are limited in the choice of the strike or maturity. On the other hand, option pricing and hedging requires advanced statistical methods and computational resources that are not available to all investors.

The second approach to portfolio insurance provides a much simpler and less restricted implementation. Indeed, floor-based strategies consist of directly adjusting the portfolio exposure over time to maintain a minimal guarantee value, referring to the floor level. These strategies are based on the *constant proportion portfolio insurance* (CPPI) allocation mechanism introduced by Black and Perold ([8]) and only differ in the design of the floor process. The CPPI strategy allocates dynamically the portfolio value between two assets: a risky asset and a risk-free asset. The floor is set at the inception of strategy and is assumed to evolve at the risk free rate. This design allows an investor to insure a proportion of his initial capital. The exposition into these assets is based on the distance of the portfolio value with its floor level, corresponding to the guarantee, and a given parameter, the so-called multiple, which measures the market exposition and can be related for example to risk aversion. The weight of the risky asset decreases or increases as the portfolio value converges towards or moves away from the floor level. Then, the initial level and adjustment speed of the exposure in the risky asset is amplified by the value of the multiple. Higher multiple value results in an higher initial exposure into the risky asset with a higher variability over time.

[1] The COVID-19 crisis and the Russia-Ukraine war.
[2] Subprime crisis, sovereign debt.

This parameter drives the investor ability to benefit from a rise in asset prices but, on the opposite side, increase the risk of reaching faster the minimal desired portfolio value. Note that it must be upper bounded to control the gap risk (i.e. the portfolio value becomes smaller than the floor).

Although the CPPI provides a flexible and easy to implement insurance strategy, it comes at the cost of two major drawbacks. Initially developed in continuous time, this strategy ensures that the portfolio value to never be lower than the floor level (i.e. there is no gap risk). However, under real market conditions, time is discrete. Thus asset prices exhibit a jump risk making this strategy subject to the gap risk ([9]). The main issues with gap risk is twofold: (i) the solution cannot guarantee a minimal portfolio value with probability one and (ii) once the portfolio breaches the floor, the portfolio is monetized (equivalently cash-lock). The exposure to the risky asset is set to zero and the portfolio can no longer benefit from any rise in asset prices. In general, the second issue does not necessarily require the portfolio to breach the floor level. The cash-lock risk occurs implicitly since, before breaching the floor, the level of the exposure to the risky asset is already significantly reduced. In most cases, the exposure mechanically becomes close to zero before the gap risk materializes. Then the portfolio becomes almost fully concentrated on risk free asset and thus misses a large part of a potential market increase.

The other major concern with the CPPI framework comes from the drawdown risk. By definition, this strategy only focuses on one aspect of the downside risk, the initial capital loss. The allocation scheme does not take into account the current gain of the portfolio and thus is subject to an high drawdown risk. Let us consider an investor with an initial capital of \$100 and floor level of \$70. If the portfolio reaches a net asset value of \$300 then the maximum possible loss for the investor is about 76.6% [3]. It is unlikely that investors will tolerate such loss level. Indeed, as suggested by Cheklov et al. ([10],[11]), investors usually withdraws their funds after a drawdown of about 20% on a one year time period.

These issues lead to several modifications of the initial CPPI framework from the choice of the multiple to the choice of the floor process. For instance, Ben Ameur and Prigent ([12]) address the gap risk and thus indirectly the cash-lock risk by allowing the multiple to vary over time. They use a risk control approach based on quantile and expected shortfall criteria to select the multiple conditionally to the market environment. They find that using conditional multiple provides significant different performance than the standard CPPI formulation due to the greater reactivity to the local market configuration. Thus the strategy benefits from low risk environment to be more inclined to increase its exposure to the risky asset and reciprocally to be more conservative in high risk environment.

Other extensions focus on the change of the floor process. One of the most known alternative to the CPPI strategy is the *time varying portfolio protection* (TIPP) strategy of Grossman and Zhou ([13]) which focus specifically on controlling the maximum drawdown. The TIPP considers the floor level as a step function increasing every time the portfolio reaches a new maximum value. The floor dynamic caps the exposure to the risky asset and thus ensures the portfolio drawdown to not exceed a predefined level. However, every time the portfolio reaches a new maximum the exposure is reset to a lower level limiting potential future gains. In a less restrictive approach, Kanniganti and Boulier ([14]) propose a more flexible framework based on two different floor process: (i) the margin and (ii) the ratchet effects. The margin effect consists in setting the initial floor higher than the target floor and use the difference, namely the margin, as a reserve to differ in time the investment mechanism. This reserve is partially or fully consumed to increase the strategy's exposure to the risky asset when it becomes too low. This reduces the risk of cash-lock. Conversely, ratchet effects increase the floor level when the strategy value increases above a predefined level. This mechanism is used to lock a proportion of the strategy's current gain and then limit the drawdown risk. However, the authors limit their work to arbitrary choices of decrease and increase of the floor level. Based on this framework, Ben Ameur and Prigent ([15]) propose for the two effects to adjust the floor level according to the same risk control they use to find conditional multiples ([12]). As a result, the floor is adjusted according to the expectation of the risk of the underlying asset. They provide, for both margin and ratchet effects, a set of rules to update the floor level while maintaining a risk control

[3] Assuming there is no gap-risk.

over the strategy.

However, in their formulation, the use of the risk control implies strong conditions. In what follows, we consider the margin based CPPI strategies. Floor adjustments subject to the risk control are triggered if the portfolio value becomes smaller than the conditional floor (equal to the target floor plus the margin) of if the cushion level is (conditionally) expected at the next period to become negative. In order to be active, this latter rule, requires either that the underlying asset must be subject to substantial losses for low to moderate multiple level or to consider very high multiple level to compensate for lower loss magnitude. Additionally, previous conditions can lead to too conservative strategy (small market exposure) since the strategy can end up in a cash-lock situation with a remaining margin since the magnitude of the expected price variation might never be enough to expect a negative cushion.

The purpose of this article is to combine the approach of Ben Ameur and Prigent ([15]) with an additional control of the minimum market exposition. Indeed, our approach is twofold : first we want to introduce a more versatile way to trigger margin effects while maintaining a local risk control over the strategy; second we search to better benefit from market rises by keeping the marker exposure above a given minimum level. in this respect, we propose an ex-post version of the triggering mechanism which relies directly on the exposure level of the strategy as in Boulier and Kanniganti ([14]). Second, we determine the floor adjustments based on a risk control which focus on the variability of the cushion instead of its level.

The paper is organized as follows. Section 2 presents the standard CPPI framework in discrete-time. Section 3 reviews the time-varying floor approach of Ben Ameur and Prigent (2018) applied to the CPPI with conditional floors when the gap risk is controlled by means of a quantile criterion. Section 4 introduces and details the margin based CPPI, especially when we impose a lower bound on the market exposure. Finally, section 5 provides the comparative analysis of our contribution using simulated and empirical data.

2 The CPPI strategy in discrete-time

In the discrete-time framework, at a set of trading dates t_k, the CPPI strategy allocates the portfolio value V_{t_k} between two assets: the risky asset S_{t_k} and the risk-free or reserve asset B_{t_k} over a given investment horizon T. The allocation mechanism consists of investing an amount called the exposure, $e_{t_k} = m * C_{t_k} = m * (V_{t_k} - P_{t_k})$ into the risky asset S_{t_k} and the remaining amount $V_{t_k} - e_{t_k}$ into the risk free asset B_{t_k}. The exposure is a function of the distance between the portfolio value V_{t_k} and the floor level P_{t_k}, namely the cushion C_{t_k} and of the multiple, $m \in \mathbb{R}^{+,*}$. The multiple can be usually related to the investor risk aversion.

In the standard formulation, the floor level is determined at inception and evolves at the same rate as the reserve asset, namely with returns, $r^B_{t_k}$, over the period $[t_{k-1}, t_k]$. For instance, an investor requiring a capital insurance of 70% at one year horizon with a risk-free rate equal to 3% per year sets at inception his floor level to $P_{t_0} = 0.7 * V_{t_0} * \exp[-0.03]$. Finally, in the case of a floor breach, i.e. $C_{t_k} \leq 0$, the exposure is immediately set to zero and the portfolio becomes fully concentrated in the risk free asset. Therefore, due to the discrete-time setting, there is a non-negligible probability that the targeted guarantee is not meet and the actual portfolio value is lower than the desired one.

The strategy dynamic is obtained through a two-step process: (i) the implementation step and (ii) the evaluation step. The first stage allocates at time t_k the portfolio value into the risky and risk-less asset while the second assesses at time t_{k+1} the results of the allocation. Therefore, we get the following representation of the CPPI strategy:

Implementation $$V_{t_k} = \frac{e_{t_k}}{S_{t_k}} * S_{t_k} + \frac{V_{t_k} - e_{t_k}}{B_{t_k}} * B_{t_k} \quad (1)$$

Evaluation $$V_{t_{k+1}} = \frac{e_{t_k}}{S_{t_k}} * S_{t_{k+1}} + \frac{V_{t_k} - e_{t_k}}{B_{t_k}} * B_{t_{k+1}} \quad (2)$$

From these two steps, we deduce the portfolio value and cushion dynamics over one period of time $[t_k, t_{k+1}]$. The portfolio dynamics is given by:

$$\Delta V_{t_{k+1}} = V_{t_{k+1}} - V_{t_k} = e_{t_k} * \frac{\Delta S_{t_{k+1}}}{S_{t_k}} + (V_{t_k} - e_{t_k}) * \frac{\Delta B_{t_{k+1}}}{B_{t_k}} \tag{3}$$

We deduce the cushion dynamics from the previous equation. Indeed, by definition, the cushion satisfies $C_{t_k} = V_{t_k} - P_{t_k}$. Thus, we have:

$$\begin{aligned}\Delta C_{t_{k+1}} = C_{t_{k+1}} - C_{t_k} &= \Delta V_{t_{k+1}} - \Delta P_{t_{k+1}} \\ &= e_{t_k} * \frac{\Delta S_{t_{k+1}}}{S_{t_k}} + (V_{t_k} - e_{t_k}) * \frac{\Delta B_{t_{k+1}}}{B_{t_k}} - \Delta P_{t_{k+1}}\end{aligned}$$

Due to the fact that $e_{t_k} = m * C_{t_k}$ and $V_{t_k} = C_{t_k} + P_{t_k}$, the previous expression becomes:

$$\begin{aligned}\Delta C_{t_{k+1}} &= m * C_{t_k} * \frac{\Delta S_{t_{k+1}}}{S_{t_k}} + (C_{t_k} + P_{t_k} - m * C_{t_k}) * \frac{\Delta B_{t_{k+1}}}{B_{t_k}} - \Delta P_{t_{k+1}} \\ &= C_{t_k} * \left(m * \frac{\Delta S_{t_{k+1}}}{S_{t_k}} + (1-m) * \frac{\Delta B_{t_{k+1}}}{B_{t_k}} \right) + P_{t_k} * \frac{\Delta B_{t_{k+1}}}{B_{t_k}} - \Delta P_{t_{k+1}}\end{aligned}$$

Since the floor P_{t_k} evolves at the same rate, $r^B_{t_{k+1}}$, as the reserve asset B_{t_k} we deduce that:

$$P_{t_k} * \frac{\Delta B_{t_{k+1}}}{B_{t_k}} - \Delta P_{t_{k+1}} = 0$$

Finally, the dynamics of the cushion is given by:

$$\Delta C_{t_{k+1}} = C_{t_k} * \left(m * \frac{\Delta S_{t_{k+1}}}{S_{t_k}} + (1-m) * \frac{\Delta B_{t_{k+1}}}{B_{t_k}} \right) \tag{4}$$

$$C_{t_{k+1}} = C_{t_k} * \left(1 + m * \frac{\Delta S_{t_{k+1}}}{S_{t_k}} + (1-m) * \frac{\Delta B_{t_{k+1}}}{B_{t_k}} \right) \tag{5}$$

If we consider that $r^B_{t_k}$ is very small (usually due to the small time period $[t_k, t_{k+1}]$), the previous equation simplifies to:

$$C_{t_{k+1}} \approx C_{t_k} * \left(1 + m * \frac{\Delta S_{t_{k+1}}}{S_{t_k}} \right) = (V_{t_k} - P_{t_k}) * \left(1 + m * \frac{\Delta S_{t_{k+1}}}{S_{t_k}} \right) \tag{6}$$

This equation fully describes the behavior of the strategy and the role of the parameters. Indeed, it appears that the multiple drives the variability of the cushion while, in some sense, the floor controls its level. Moreover, this expression provides additional useful information over the relationship between the parameters choice and the strategy risks. For instance, one way to escape rapidly from the cash-lock risk (i.e. C_{t_k} close to zero) is to use a high enough multiple value. However, such value increases the risk of breaching the floor level (i.e. increases the gap risk). Alternatively, the floor can be adjusted, downwards or upwards, to either mitigate the cash-lock risk or the drawdown risk, respectively. A lower floor level mechanically results in an higher cushion and thus exposure, while a higher floor level reduces the exposure and set a lower tolerance for losses.

In what follows, we are going to consider various time varying and conditional floors. However, we note the following property of "independence" w.r.t. the floor.

Remark 1 *(Cushion positivity and floor) According to Equation 6, the positivity of the cushion after the variations of the asset prices does not depend on the floor value.*

3 Time varying floor framework

Due to these relationships, Ben Ameur and Prigent ([12],[15]) use the previous equation (5) as a starting point to provide a risk based framework to the selection of the parameters. In a first instance, they show in ([12]) that the gap risk can be controlled when considering multiples satisfying the following quantile rule:

$$\mathbb{P}\left(\forall\ t_k \in [0,T], C_{t_k} > 0\right) \geq 1-\epsilon \underset{r_{t_k}<<}{\Leftrightarrow} \mathbb{P}\left(\forall\ t_k \in [0,T], (1+m*\frac{\Delta S_{t_k}}{S_{t_{k-1}}}) > 0\right) \geq 1-\epsilon$$

with $\epsilon \in (0,1)$. Equivalently, considering $M_T = \max_{1\leq l\leq n}\left[-\frac{\Delta S_{t_l}}{S_{t_l-1}}\right]$ the maximum loss over one period of time, we get:

$$\begin{aligned}
\mathbb{P}\left(\forall\ t_k \in [0,T], -\frac{\Delta S_{t_l}}{S_{t_{l-1}}} < \frac{1}{m}\right) \geq 1-\epsilon \Leftrightarrow \mathbb{P}\left(M_T < \frac{1}{m}\right) &\geq 1-\epsilon \\
\Leftrightarrow F_{M_T}\left(\frac{1}{m}\right) &\geq 1-\epsilon \\
\Leftrightarrow \frac{1}{m} &\geq F^{-1}_{M_T}(1-\epsilon) \\
\Leftrightarrow m &< \frac{1}{F^{-1}_{M_T}(1-\epsilon)}
\end{aligned}$$

where $F^{-1}_{M_T}$ is the inverse of the cumulative distribution function of M_T. This approach allows investors to target multiple value depending on their choice of the probability threshold ϵ. Indeed, the upper bound is an increasing function of this latter one. For example, in some sense, a risk averse investor will choose a small ϵ implying that he will select a low multiple. However, this rule considers the asset returns distribution in its globality and do not account for its temporal properties. In this way, the multiple is constant over the entire investment period and thus the strategy cannot adapt to the different risk environments.

In this context, the authors address the lack of adaptability by considering a more general framework. First the multiple is no longer constant. Second it evolves over time in such a way that the gap risk is controlled over two consecutive trading dates. This local feature yields from the use of the current state of the cushion in the risk control selection of the parameter. Since the cushion is mainly driven by the asset returns, this approach allows to account for the asset price dynamic and thus its different risk environments.

This framework is not only limited to the selection of the multiple under a gap risk control. Ben Ameur and Prigent ([15]), extended the previous approach to the floor process. Based on the previous work of Kanniganti and Boulier ([14]), they show that the floor can be adjusted to reduce both the cash-lock risk and the drawdown risk, using respectively margin and ratchet effects, while maintaining a gap risk control. The margin effect consists of reducing the floor level to regain in exposure into the risky asset. Reciprocally, the ratchet effect increases the floor level to lock in the current gains of the strategy. Both of these effects are triggered based on predefined events corresponding to specific states of the strategy. For example, in the case of the margin effect the floor can be reduced when the exposure decreases below a specific level.

The time-varying floor mechanism is common to both effects. First it assumes a target floor, denoted $\hat{P}_{t_k}$, referring to the usual floor of the standard strategy. This floor allows to control the global loss risk of the portfolio over time and allows to recover a predefined percentage of the initial investment amount at the terminal horizon. If at any trading date, t_k, the portfolio breached the target floor (i.e. $\hat{C}_{t_k} = V_{t_k} - \hat{P}_{t_k} \leq 0$) then the portfolio becomes monetized. Second this mechanism allows the investor to modify his floor at any time during the management period. Thus it defines an effective dynamic and conditional floor as follows:

$$\nabla P_{t_k} = P^+_{t_k} - P^-_{t_k}, \tag{7}$$

which means that ∇P_{t_k} represents the variation of the floor at time t_k due to the specific choice of the new floor $P^+_{t_k}$.

- The value $P^-_{t_k}$ is equal to the previous floor value chosen at time t_{k-1} for the period $[t_{k-1}, t_k[$ and invested in the riskless asset with rate $r^B_{t_k}$ during this time period. Thus it evolves according to:

$$P^-_{t_k} = P^+_{t_{k-1}} * \exp(r^B_{t_k}).$$

- The value $P^+_{t_k}$ is chosen at time t_k in order to satisfy the portfolio management objectives at that time. This can be based on a triggering event modeled by a Bernoulli random variable X_{t_k} depending on the considered effects.

In the same way, we define the variations of the cushion at time t_k, resulting from the choice of the portfolio strategy at time t_k:

$$\nabla C_{t_k} = C^+_{t_k} - C^-_{t_k}.$$

Note that we have $P^+_{t_k} \geq \hat{P}_{t_k}$. Therefore, we get the following general form for the dynamic floor $P^+_{t_k}$:

Proposition 2 *(Choice of the new floor) At any time t_k, the floor $P^+_{t_k}$ is chosen in the following manner:*

$$P^+_{t_k} = \begin{cases} h(t_k, \Gamma_{t_k}) & \text{if } X_{t_k} = 1, \\ \\ P^-_{t_k} = P^+_{t_{k-1}} * (1 + r^B_{t_k}) & \text{if } X_{t_k} = 0, \end{cases} \tag{8}$$

where $h(t_k, \Gamma_{t_k})$ denotes a generic function and Γ a set of parameters fully determined from the considered effects.

Based on this new floor process, we deduced the following dynamics for the strategy value and the cushion level:

$$V_{t_{k+1}} = V_{t_k} + e^+_{t_k} * \frac{\Delta S_{t_{k+1}}}{S_{t_k}} + (V_{t_k} - e^+_{t_k}) * \frac{\Delta B_{t_{k+1}}}{B_{t_k}} \tag{9}$$

with $e^+_{t_k} = m * C^+_{t_k} = m * (V_{t_k} - P^+_{t_k})$ and the dynamics of the cushion is defined by:

$$C^-_{t_{k+1}} \approx C^+_{t_k} * \left(1 + m * \frac{\Delta S_{t_{k+1}}}{S_{t_k}}\right) = \left(V_{t_k} - P^+_{t_k}\right) * \left(1 + m * \frac{\Delta S_{t_{k+1}}}{S_{t_k}}\right) \tag{10}$$

3.1 Value-at-Risk constraints on the cushion value

The floor can be adjusted up or down based on the following risk control on the cushion value:

$$\forall\ k \in \mathbb{N},\ \mathbb{P}^{\mathcal{G}_{t_k}}(C^-_{t_{k+1}} < -L_{t_k}) < \epsilon \tag{11}$$

where $\forall\ k \in \mathbb{N}, L_{t_k} > 0$ is a predefined threshold and $\mathcal{G}_{t_k}$ corresponds to a set of information such that $\mathcal{F}_{t_{k-1}} \subset \mathcal{G}_{t_k}$ with $\mathcal{F}_{t_k} = \sigma\left(\frac{\Delta S_{t_1}}{S_{t_0}}, \ldots, \frac{\Delta S_{t_k}}{S_{t_{k-1}}}\right)$ the σ-algebra generated by the asset returns. This quantile condition is considered as "local" since the control at time t_k concerns only the variation on the time period $]t_k, t_{k+1}]$.

Developing this risk control results to the following restriction over the floor level:

$$\begin{aligned} C^-_{t_{k+1}} < -L_{t_k} &\Leftrightarrow C^+_{t_k} * \left(1 + m * \frac{\Delta S_{t_{k+1}}}{S_{t_k}}\right) < -L_{t_k} \\ &\Leftrightarrow \frac{\Delta S_{t_{k+1}}}{S_{t_k}} < -\frac{1}{m} * \left(1 + \frac{L_{t_k}}{C^+_{t_k}}\right) \end{aligned}$$

Let $F_{\frac{\Delta S_{t_{k+1}}}{S_{t_k}}}(\cdot)$ be the conditional cumulative distribution function of the asset returns w.r.t. the information $\mathcal{G}_{t_k}$. We assume that it is invertible.[4]

$$\begin{aligned}
\forall\ k \in \mathbb{N},\ \mathbb{P}^{\mathcal{G}_{t_{k-1}}}(C^-_{t_{k+1}} < -L_{t_k}) < \epsilon &\Leftrightarrow F_{\frac{\Delta S_{t_{k+1}}}{S_{t_k}}}\left(-\frac{1}{m}\left(1+\frac{L_{t_k}}{C^+_{t_k}}\right)\right) < \epsilon \\
&\Leftrightarrow -\frac{1}{m}\left(1+\frac{L_{t_k}}{C^+_{t_k}}\right) < F^{-1}_{\frac{\Delta S_{t_{k+1}}}{S_{t_k}}}(\epsilon) \\
&\Leftrightarrow -\frac{L_{t_k}}{C^+_{t_k}} < \left(1 + m * F^{-1}_{\frac{\Delta S_{t_{k+1}}}{S_{t_k}}}(\epsilon)\right)
\end{aligned}$$

Denote by $\theta^m_{t_k}(\epsilon)$ the term $\left(1 + m * F^{-1}_{\frac{\Delta S_{t_{k+1}}}{S_{t_k}}}(\epsilon)\right)$ which is the quantile of $\left(1 + m * \frac{\Delta S_{t_{k+1}}}{S_{t_k}}\right)$ at the probability level ϵ. Therefore we deduce:

- If $\theta^m_{t_k}(\epsilon) < 0$ then

$$\begin{aligned}
-\frac{L_{t_k}}{\theta^m_{t_k}(\epsilon)} > C^+_{t_k} &\Leftrightarrow -\frac{L_{t_k}}{\theta^m_{t_k}(\epsilon)} > (V_{t_k} - P_{t_k^+}) \\
&\Leftrightarrow V_{t_k} + \frac{L_{t_k}}{\theta^m_{t_k}(\epsilon)} < P_{t_k^+}
\end{aligned}$$

  and since $V_{t_k} - P^+_{t_k} > 0$ we have the following restriction over the choice of the new floor level:

$$V_{t_k} + \frac{L_{t_k}}{\theta^m_{t_k}(\epsilon)} < P_{t_k^+} < V_{t_k} \tag{12}$$

- Finally if $\theta^m_{t_k}(\epsilon) > 0$ we have

$$P_{t_k^+} < \min\left[V_{t_k} + \frac{L_{t_k}}{\theta^m_{t_k}(\epsilon)}, V_{t_k}\right] \tag{13}$$

  Since $\frac{L_{t_k}}{\theta^m_{t_k}(\epsilon)} > 0$ then $P_{t_k^+} < V_{t_k}$. In this configuration the underlying risk is low enough to not impose any particular restriction on the choice of the floor level.

Proposition 3 *(VaR constraints on the new floor due to the risk control on the cushion value) The risk control adjustment of the floor is completely determined by the sign of the quantity $\theta^m_{t_k}(\epsilon)$:*

1. *If $\theta^m_{t_k}(\epsilon) < 0$ then*

$$V_{t_k} + \frac{L_{t_k}}{\theta^m_{t_k}(\epsilon)} < P_{t_k^+} < V_{t_k} \tag{14}$$

2. *If $\theta^m_{t_k}(\epsilon) > 0$, then*

$$P_{t_k^+} < V_{t_k} \tag{15}$$

As emphasized in previous proposition, the sign of the quantile $\theta^m_{t_k}(\epsilon)$ plays a key role when controlling locally the cushion value.

[4] Otherwise, we consider its left inverse, as it is a monotononic function.

Remark 4 *The quantile $\theta^m_{t_k}(\epsilon)$ depends on the conditional distribution of the asset returns as follows:*

$$\theta^m_{t_k}(\epsilon) < 0 \Leftrightarrow F^{-1}_{\frac{\Delta S_{t_{k+1}}}{S_{t_k}}}(\epsilon) < -\frac{1}{m}$$

$$\theta^m_{t_k}(\epsilon) > 0 \Leftrightarrow F^{-1}_{\frac{\Delta S_{t_{k+1}}}{S_{t_k}}}(\epsilon) > -\frac{1}{m}$$

For example, if $m = 3$ then the expected asset return over $[t_k, t_{k+1}]$ at a given probability level $F^{-1}_{\frac{\Delta S_{t_{k+1}}}{S_{t_k}}}(\epsilon)$ must be lower than -33.33%. This anticipated loss threshold is very significant and suggests that the underlying instrument must be in very bad configuration to reach such loss level. Likewise, if the expected asset returns is above this level then the instrument is considered to be in a good enough configuration to require any risk control. The multiple plays an important role in the classification of the underlying risk environment. Indeed, the multiple determines the sensitivity of the strategy to the risky asset. Thus a low multiple implies a lower sensitivity and results in an higher loss threshold. Reciprocally, strategies with higher multiples are more prone to require a risk control since there more sensitive to the risky asset and thus considered riskier. Therefore, the multiple determines the definition of the risk environment.

To summarize, the choice of the new floor $P^+_{t_k}$ at time t_k, based on the risk control of the cushion value, can be expressed as follows.

Proposition 5 *(Choice of the new floor with risk control of the cushion value) At any time t_k, we follow the following process:*

1. *If $\widehat{P}_{t_k} \geq V_{t_k}$, then the exposure is set to 0 until maturity.*
2. *If $\widehat{P}_{t_k} < V_{t_k}$, then the floor $P^+_{t_k}$ is chosen in the following manner:*

$$P^+_{t_k} = \begin{cases} h(t_k, \Gamma_{t_k}) & \text{if } X_{t_k} = 1, \\ P^-_{t_k} = P^+_{t_{k-1}} * (1 + r^B_{t_k}) & \text{if } X_{t_k} = 0, \end{cases} \tag{16}$$

with

$$X_{t_k} = \begin{cases} 1 & \text{if } P^-_{t_k} > V_{t_k} \text{ or } P^-_{t_k} < V_{t_k} \text{ but } \theta^m_{t_k}(\epsilon) < 0 \\ 0 & \text{if } P^-_{t_k} < V_{t_k} \text{ and } \theta^m_{t_k}(\epsilon) > 0 \end{cases} \tag{17}$$

and

$$h(t_k, \Gamma_{t_k}) = \begin{cases} \widehat{P}_{t_k} + q_{t_k} * \widehat{C}_{t_k} \text{ with } 0 < q_{t_k} < \dfrac{(V_{t_k} - \widehat{P}_{t_k})}{\widehat{C}_{t_k}} & \text{if } P^-_{t_k} > V_{t_k} \\ \left(V_{t_k} + \dfrac{L_{t_k}}{\theta^m_{t_k}(\epsilon)}\right) & \text{if } P^-_{t_k} < V_{t_k} \text{ and } \theta^m_{t_k}(\epsilon) < 0 \end{cases} \tag{18}$$

Note that this latter choice corresponds to the choice of the maximal possible exposure when $P^-_{t_k} < V_{t_k}$ and $\theta^m_{t_k}(\epsilon) < 0$, under the quantile constraint.

3.2 Alternative risk control

As discussed previously, the risk control is not easy to apply since it requires very strong conditions to be active. In this context, we can control for example the downside variation of the cushion induced by the choice of a new floor level instead of its direct level. Such approach allows to remove the dependence of the risk control activation to the multiple. Thus, the application of the risk control is now fully driven by the distribution of the asset returns. Let $\Delta C^-_{t_{k+1}} = C^-_{t_{k+1}} - C^+_{t_k}$ be the variation of the cushion right on the time period $]t_k, t_{k+1}]$. The risk control is design as follows:

$$\forall\, k \in \mathbb{N},\ \mathbb{P}^{\mathcal{G}_{t_k}}\left(\Delta C^{-}_{t_{k+1}} < -\tilde{L}_{t_k}\right) < \epsilon \tag{19}$$

Since we have:

$$\Delta C^{-}_{t_{k+1}} = C^{-}_{t_{k+1}} - C^{+}_{t_k} \approx C^{+}_{t_k} * m * \frac{\Delta S_{t_{k+1}}}{S_{t_k}},$$

we deduce that

$$\begin{aligned}\forall\, k \in \mathbb{N},\ \mathbb{P}^{\mathcal{G}_{t_k}}\left(\Delta C^{-}_{t_{k+1}} < -\tilde{L}_{t_k}\right) < \epsilon &\Leftrightarrow \mathbb{P}^{\mathcal{G}_{t_k}}\left(m * C^{+}_{t_k} * \frac{\Delta S_{t_{k+1}}}{S_{t_k}} < -\tilde{L}_{t_k}\right) < \epsilon \\ &\Leftrightarrow \mathbb{P}^{\mathcal{G}_{t_k}}\left(\frac{\Delta S_{t_{k+1}}}{S_{t_k}} < \frac{-\tilde{L}_{t_k}}{m * C^{+}_{t_k}}\right) < \epsilon\end{aligned}$$

Assuming that the cumulative conditional distribution function of the asset returns, $F_{\frac{\Delta S_{t_{k+1}}}{S_{t_k}}}(\cdot)$ is invertible and $\tilde{\theta}^{m}_{t_k}(\epsilon) = m * F^{-1}_{\frac{\Delta S_{t_{k+1}}}{S_{t_k}}}(\epsilon)$, we obtain:

$$\begin{aligned}\forall\, k \in \mathbb{N},\ \mathbb{P}^{\mathcal{G}_{\sqcup_\|}}\left(\Delta C^{-}_{t_{k+1}} < -\tilde{L}_{t_k}\right) < \epsilon &\Leftrightarrow \frac{-L_{t_k}}{C^{+}_{t_k}} < \tilde{\theta}^{m}_{t_k}(\epsilon) \\ &\Leftrightarrow -\tilde{L}_{t_k} < C^{+}_{t_k} * \tilde{\theta}^{m}_{t_k}(\epsilon) \\ &\Leftrightarrow -\tilde{L}_{t_k} < \left(V_{t_k} - P^{+}_{t_k}\right) * \tilde{\theta}^{m}_{t_k}(\epsilon)\end{aligned}$$

Finally based on the sign of $\tilde{\theta}^{m}_{t_k}(\epsilon)$ we have the following relationships:

$$\begin{cases} \tilde{\theta}^{m}_{t_k}(\epsilon) < 0 \implies V_{t_k} + \dfrac{\tilde{L}_{t_k}}{\tilde{\theta}^{m}_{t_k}(\epsilon)} \le P^{+}_{t_k} \\ \\ \tilde{\theta}^{m}_{t_k}(\epsilon) > 0 \implies P^{+}_{t_k} \le V_{t_k} + \dfrac{\tilde{L}_{t_k}}{\tilde{\theta}^{m}_{t_k}(\epsilon)}, \end{cases} \tag{20}$$

the latter condition being always satisfied by construction of $P^{+}_{t_k}$.

Remark 6 *These inequalities only differ from the ones (12, 13) obtained with the previous gap risk control on how the risk environment is determined.*

$$\textbf{\textit{Level}} \qquad \theta^{m}_{t_k}(\epsilon) = 1 + m * F^{-1}_{\frac{\Delta S_{t_{k+1}}}{S_{t_k}}}(\epsilon) \tag{21}$$

$$\textbf{\textit{Variation}} \qquad \tilde{\theta}^{m}_{t_k}(\epsilon) = m * F^{-1}_{\frac{\Delta S_{t_{k+1}}}{S_{t_k}}}(\epsilon) \tag{22}$$

Looking at these measures, we note that the use of the new risk control provides two additional features: (i) the risk environment (i.e. the sign of θ) is independent from the multiple m and (ii) the risk measure is no longer restricted to extreme risk scenario. For the special cases $L_{t_k} = 0$ and $\tilde{L}_{t_k} = 0$, the first risk is to get a negative floor (gap risk of the conditional floor) while the second risk corresponds to a (simple) decrease of the cushion.

4 Floor adjustments with margin effects

In the case of margin effects, the effective floor process is composed as the sum of two elements: (i) the target floor, $\hat{P}_{t_k}$, and (ii) the margin, $M_{t_k} > 0$. The margin works as a buffer that decreases every time a triggering event is reached. By construction, the floor is initially higher than the target floor and converges toward it gradually.

To detail the margin strategy, we first consider the case of the risk control according to previous quantile condition.

4.1 Choice of the new margin with risk control of the cushion value

A) If the portfolio value V_{t_k} satisfies $P_{t_k}^- < V_{t_k}$ then we reduce possibly the margin according to the following rule using the VaR condition, namely:

1. If $\theta_{t_k} < 0$, then the new value $P_{t_k}^+$ of the floor is equal to the usual floor plus the previous margin evaluated at time t_k, which is reduced by factor γ_{t_k}. We have:

$$P_{t_k}^+ = \hat{P}_{t_k} + M_{t_k}^+ \text{ and } M_{t_k}^+ = M_{t_k}^- \gamma_{t_k}.$$

The new cushion is equal to:

$$C_{t_k}^+ = V_{t_k} - P_{t_k}^+$$

Thus, to satisfy the general condition determining the lower bound on the floor if $\theta_{t_k} < 0$, we must set:

$$\frac{V_{t_k} + \frac{L_{t_k}}{\theta_{t_k}} - P_{t_k}^+}{M_{t_k}^-} \leq \gamma_{t_k} \leq \frac{V_{t_{k1}} - P_{t_k}^+}{M_{t_k}^-}. \tag{23}$$

In our illustrations, we set:

$$\gamma_{t_k} = \gamma_{t_k}^{VaR*} = \max\left[\left(\frac{V_{t_k} + \frac{L_{t_k}}{\theta_{t_k}} - P_{t_k}^+}{M_{t_k}^-}\right), 0\right] \tag{24}$$

2. If $\theta_{t_{k-1}} > 0$, then the VaR condition is not stringent. We keep the same floor (i.e. $P_{t_k}^+ = P_{t_k}^+$).

B) If the portfolio value V_{t_k} satisfies $P_{t_k}^- > V_{t_k}$ then we reduce the floor as follows: We define a new margin $M_{t_k}^+$ equal to a given proportion q_{t_k} $(0 < q_{t_k} < 1)$ of the target cushion $\widehat{C}_{t_k} = V_{t_k} - \widehat{P}_{t_{k1}}$. Thus we set:

$$P_{t_k}^+ = \widehat{P}_{t_k} + M_{t_k}^+ \text{ with } M_{t_k}^+ = q_{t_k} * \left(V_{t_k} - \widehat{P}_{t_k}\right).$$

Recall that, if $\widehat{P}_{t_{k1}} \geq V_{t_k}$, then the exposure is set to 0 until maturity.

Remark 7 *In the original case of the margin introduced by Boulier and Kanniganti (2005), the proportion $\gamma_{t_k} = q_{t_{k-1}}$ is assumed to be constant. Additionally, there is no explicit risk control. In our framework, this proportion is variable and based on the quantile condition depending at each time on the values of several parameters such as m, L_t, and $\widehat{P}_0$ together with the current portfolio value V_{t_k}.*

To summarize, for the margin based strategy based on risk control, the floor process $(P_{t_k}^+)_k$ is defined through a sequence of margins $(M_{t_l})_l$ as follows.

Proposition 8 *(Choice of the new margin with risk control of the cushion value) At any time t_k, the process $P_{t_k}^+$ is defined as follows:*

1. *If $\widehat{P}_{t_k} \geq V_{t_k}$, then the exposure is set to 0 until maturity.*

2. *If $\widehat{P}_{t_k} < V_{t_k}$, then the floor $P^+_{t_k}$ satisfies:*

$$P^+_{t_k} = \begin{cases} h_m(t_k, \Gamma_{t_k}) & \text{if } X_{t_k} = 1, \\ \\ P^-_{t_k} = P^+_{t_{k-1}} * (1 + r^B_{t_k}) & \text{if } X_{t_k} = 0, \end{cases} \tag{25}$$

with

$$X_{t_k} = \begin{cases} 1 & \text{if } P^-_{t_k} > V_{t_k} \text{ or } P^-_{t_k} < V_{t_k} \text{ but } \theta^m_{t_k}(\epsilon) < 0 \\ \\ 0 & \text{if } P^-_{t_k} < V_{t_k} \text{ and } \theta^m_{t_k}(\epsilon) > 0 \end{cases} \tag{26}$$

and

$$h_m(t_k, \Gamma_{t_k}) = \begin{cases} \widehat{P}_{t_k} + M^+_{t_k} \text{ with } M^+_{t_k} = q_{t_k}\widehat{C}_{t_k} \text{ and } 0 < q_{t_k} < 1 & \text{if } P^-_{t_k} > V_{t_k} \\ \\ \widehat{P}_{t_k} + M^+_{t_k} \text{ with } M^+_{t_k} = M^-_{t_k} * \gamma^{VaR*}_{t_k} & \text{if } P^-_{t_k} < V_{t_k} \text{ and } \theta^m_{t_k}(\epsilon) < 0 \end{cases} \tag{27}$$

Finally from these conditions, the portfolio manager is able to reduce the cash lock risk while controlling the gap risk. For instance, when $\theta^m_{t_k}(\epsilon) < 0$ selecting the lower bound minimizes the cash-lock risk since it represents the greatest increase in exposure while maintaining the gap risk under control. Reciprocally, when $\theta^m_{t_k}(\epsilon) > 0$ the risk of the underlying asset is considered low enough to be subject to the risk control.

4.2 Floor adjustments with margin effects and minimal exposure

As seen previously, the dependence of the triggering event to $\theta^m_{t_k}(\epsilon)$ limits drastically its reachability, except for very small values of the probability threshold φ. As a result, the strategy based on the previous risk control is almost identical to the one introduced by Boulier and Kanniganti ([14]). Our approach to mitigate this dependency is to change the triggering event for a simpler one only based on the observed exposure of the strategy. In what follows, instead of monitoring an expected breach of the running cushion, we focus on the proportion invested in the risky asset defined as:

$$w_{t_k} = \frac{e_{t_k}}{V_{t_k}} = \frac{m * C_{t_k}}{V_{t_k}}.$$

This choice is equivalent to monitor the cushion due to their proportional relationship[5] (note also that $\text{sgn } w^-_{t_k} = \text{sgn } C^-_{t_k}$) but provides an easier interpretation. Let $\overline{w}_{t_k} \in \mathbb{R}^+$ be the triggering threshold such that a triggering event occurs every time $w_{t_k} \leq \overline{w}_{t_k}$. This event is considered as ex-post compared to the previous one since it depends only on a realized observation and not on any expectation. Thus the

$$X_{t_k} = 1_{\{w^-_{t_k} \leq \overline{w}_{t_k}\}}$$

The use of this type of triggers requires to distinguish two cases depending on the value of the multiple m and the trigger threshold $\overline{w}_{t_k}$. In the case of $m \leq \overline{w}_{t_k}$ margin call will constantly occurs since the maximal exposure is limited to a lower level than the trigger threshold:

$$\max_{1 \leq k \leq n} w^-_{t_k} = \max_{1 \leq k \leq n} \left[m * \left(1 - \frac{P^-_{t_k}}{V_{t_k}}\right)\right] < \max_{1 \leq k \leq n} \left[\overline{w}_{t_k} * \left(1 - \frac{P^-_{t_k}}{V_{t_k}}\right)\right] < \max_{1 \leq k \leq n} \overline{w}_{t_k}$$

However this case is rarely implemented in practice since in most cases portfolio managers do not have the ability to use large leverages. For instance, if $m = 3$ the portfolio exposure into the risky asset must reached at least 300% to trigger a margin effect. Moreover to benefit from a convex payoff the multiple tends to be quite high compared to the allowed level of leverage. In the case $m > \overline{w}_{t_k}$,

[5] When using the exposition level, there is no need to screen the cushion and reciprocally.

the use of soft triggers implies that the available margin at the time of the event is implicitly higher than the one initially defined. Indeed, at a triggering time we have:

$$w_{t_k} \leq \overline{w}_{t_k} \Leftrightarrow m * \left(V_{t_k} - P^-_{t_k}\right) \leq \overline{w}_{t_k} * V_{t_k} \Leftrightarrow V_{t_k} \leq P^-_{t_k} * \left(1 - \frac{\overline{w}_{t_k}}{m}\right)^{-1} = P^{(I,-)}_{t_k},$$

where $P^{(I,-)}_{t_k}$ denotes an implied effective floor from which we deduce the following implied margin:

$$M^{(I,-)}_{t_k} = P^{(I,-)}_{t_k} - \hat{P}_{t_k} = P^-_{t_k} * \left(1 - \frac{\overline{w}_{t_k}}{m}\right)^{-1} - \hat{P}_{t_k} = M^-_{t_k} + P^-_{t_k} * \frac{\overline{w}_{t_k}}{m - \overline{w}_{t_k}}$$

Thus, to set the weight above its minimal value, we must choose the new floor such as:

$$P^+_{t_k} \leq \frac{m - \overline{w}_{t_k}}{m} * V_{t_k}.$$

However, since we must choose the new floor above the target floor, we consider finally:

$$P^+_{t_k} = \max\left[\left(\frac{m - \overline{w}_{t_k}}{m} * V_{t_k}\right), \hat{P}_{t_k}\right]$$

When $\frac{m-\overline{w}_{t_k}}{m} * V_{t_k} \geq \hat{P}_{t_k}$, we note that:

$$P^+_{t_k} = \hat{P}_{t_k} + M^+_{t_k}$$

$$\text{with } M^+_{t_k} = \left(1 - \frac{\overline{w}_{t_k}}{m}\right) V_{t_k} - \hat{P}_{t_k} = \left(\hat{C}_{t_k} - \frac{\overline{w}_{t_k}}{m} V_{t_k}\right).$$

To summarize, for the margin based strategy with minimal exposure, the floor process $(P^+_{t_k})_k$ is defined through a sequence of margins $(M_{t_l})_l$ as follows.

Proposition 9 *(Choice of the new margin with minimal exposure) At any time t_k, the process $P^+_{t_k}$ is defined as follows:*

1. *If $\widehat{P}_{t_k} \geq V_{t_k}$, then the exposure is set to 0 until maturity.*
2. *If $\widehat{P}_{t_k} < V_{t_k}$, then the floor $P^+_{t_k}$ satisfies:*

$$P^+_{t_k} = \begin{cases} h_e(t_k, \Gamma_{t_k}) & \text{if } X_{t_k} = 1, \\ P^-_{t_k} = P^+_{t_{k-1}} * (1 + r^B_{t_k}) & \text{if } X_{t_k} = 0, \end{cases} \tag{28}$$

with

$$X_{t_k} = \begin{cases} 1 & \text{if } P^-_{t_k} > V_{t_k} \text{ or } P^-_{t_k} < V_{t_k} \text{ but } w_{t_k} \leq \overline{w}_{t_k} \\ 0 & \text{if } P^-_{t_k} < V_{t_k} \text{ and } w_{t_k} \geq \overline{w}_{t_k} \end{cases} \tag{29}$$

and

$$h_e(t_k, \Gamma_{t_k}) = \begin{cases} \widehat{P}_{t_k} + M^+_{t_k} \text{ with } M^+_{t_k} = q_{t_k} * \widehat{C}_{t_k} \text{ and } 0 \leq q_{t_k} \leq 1 & \text{if } P^-_{t_k} > V_{t_k} \\ \widehat{P}_{t_k} + M^+_{t_k} \text{ with } M^+_{t_k} = \max\left[\left(\hat{C}_{t_k} - \frac{\overline{w}_{t_k}}{m} * V_{t_k}\right), 0\right] & \text{if } P^-_{t_k} < V_{t_k} \text{ and } w_{t_k} \leq \overline{w}_{t_k} \end{cases} \tag{30}$$

If we impose also the constraint of minimal exposure when $P^-_{t_k} > V_{t_k}$, then we choose q_{t_k} such that $q_{t_k} = \max\left[\left(1 - \frac{\overline{w}_{t_k}}{m} \frac{V_{t_k}}{\hat{C}_{t_k}}\right), 0\right]$. Thus, the margin $M^+_{t_k}$ satisfies:

$$M^+_{t_k} = \gamma_{e,t_k} * M^-_{t_k} \text{ with } \gamma_{e,t_k} = \begin{cases} \max\left[\frac{\left(\hat{C}_{t_k} - \frac{\overline{w}_{t_k}}{m} * V_{t_k}\right)}{M^-_{t_k}}, 0\right] & \text{if } X_{t_k} = 1, \\ 1 & \text{if } X_{t_k} = 0, \end{cases} \tag{31}$$

Remark 10 *Usually the CPPI strategy is capped, meaning that the market exposure is smaller than a given percentage $\lambda \in]0,1]$ of the portfolio value (we take $\lambda = 1$ in our numerical illustrations). We have:*

$$e_{t_k} = Min\left(mC_{t_k}, \lambda V_{t_k}\right).$$

Then $e_{t_k} = mC_{t_k}$ implies that:

$$P^+_{t_k} \geq \left(\frac{m-\lambda}{m}\right) V_{t_k}.$$

Therefore, we must have:

$$\left(\frac{m-\lambda}{m}\right) V_{t_k} \leq P^+_{t_k} \leq \left(\frac{m-\overline{w}_{t_k}}{m}\right) V_{t_k}.$$

Since usually $\overline{w}_{t_k} \leq 0.3$ and $\lambda \geq 0.9$, the set of floor values $P^+_{t_k}$ satisfying the previous inequalities is not empty .

5 Numerical analysis

In this section, we compare the standard CPPI and the two different margin mechanisms previously introduced. Namely, the ex-ante mechanism which decreases the running floor every time the cushion or the expected cushion becomes negative and the ex-post mechanism based on the strategy's exposure. Then for both of these strategies we apply the same level risk control (RC) with the following threshold form:

$$L_{t_k} = \beta * \hat{C}_{t_k} \text{ with } \beta \in [0,1]$$

This choice provides an interpretation of the risk control in term of the overall risk budget and aims to restrain by how much of the later is consumed in an unfavorable configuration. Moreover, the strategies aim to target a minimal level of exposure thus we design the margin decay, p_{t_k}, when no risk control is apply such that the exposure of the strategy is reset to the predefined level $\overline{w}_{t_k}$. We obtain the following form of the parameter:

$$p_{t_k} = \max\left[0, \min\left[1, \left(\frac{V_{t_k} * \left(1 - \frac{\overline{w}_{t_k}}{m}\right) - \hat{P}_{t_k}}{M^-_{t_k}}\right)\right]\right]$$

The considered strategies aim to maintain a minimal exposure level with respect to the available margin. This exposure level is not always satisfy since when the risk control is applied the portfolio manager considers that managing the gap risk prevails over maintaining a minimal exposure level.

Finally, we use the empirical quantile over a rolling window of two years (i.e. 104 weeks) as an estimator of the conditional quantile of the underlyer returns. And we consider the set of parameters below to define the baseline strategies:

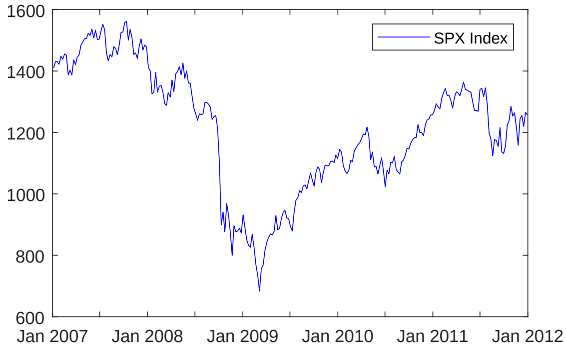
Floor guarantee	**Floor margin**	**Multiple** m	**Trigger** $\overline{w}_{t_k}$	**Threshold** β	**Quantile** ϵ	**Leverage** $\max(w_{t_k})$
15%	7.5%	6	5%	5%	1%	100%

Table 1: Reference strategy parameters

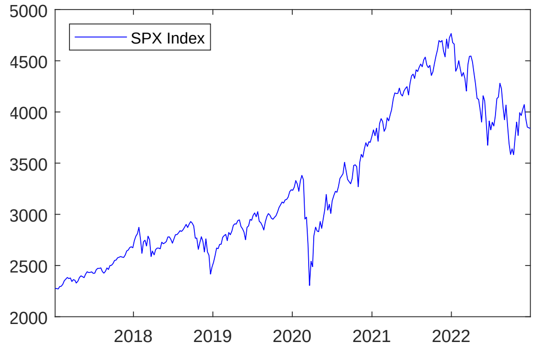
The numerical analysis is conducted in two parts: (i) the first part uses bootstrap simulation to study the strategies payoff profile and their sensitivities to parameters while (ii) the second part focus on empirical observations to illustrate their overall performances based on real market conditions. In both cases we use end of week prices[6] from the S&P500 and the Euro Stoxx 50 over two specific time periods. The first period lies from 31/12/2007 to 31/12/2011 and the second period spans from

[6] End of week = Friday

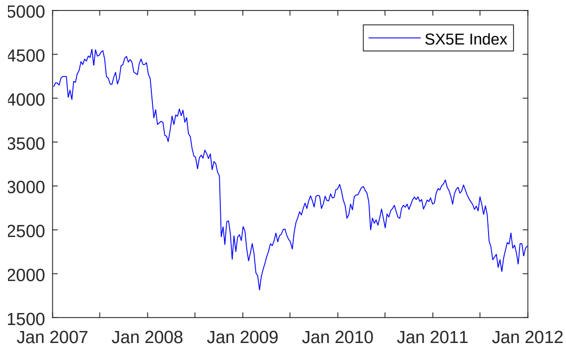
31/12/2017 to 31/12/2022. These periods are selected for their different crisis behavior (see figure 1 and table 2). For instance, the subprimes crisis defines a significant cumulative losses at a relative moderate pace followed by a slow recovery period for the S&P500 and a no recovery for the Euro Stoxx 50. Conversely, the COVID-19 crisis characterizes a very fast sell-off and accompanied by a strong and fast recovery for the S&P500 and a more moderate one for the Euro Stoxx 50.



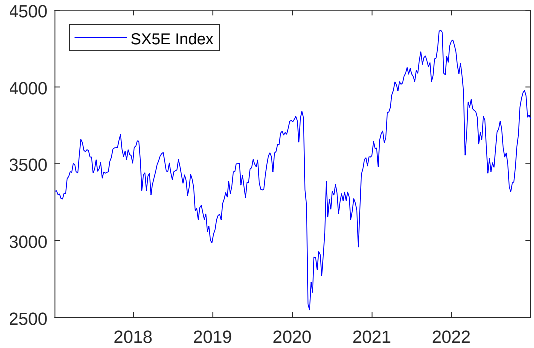
(a) SPX Index from 31/12/2006 to 31/12/2011



(b) SPX Index from 31/12/2016 to 31/12/2022



(c) SX5E Index from 31/12/2006 to 31/12/2011



(d) SX5E Index from 31/12/2016 to 31/12/2022

Figure 1: Price index of the S&P 500 Index and the Euro Stoxx 50 Index over the two considered time period on a weekly basis.

Index	**Start date**	**Drawdown date**	**Recovery date**	**Time to drawdown**	**Time to recover**	**Drawdown**	**Recover**	**Sell-off Recovery**
SPX	12-Oct-2007	06-Mar-2009	28-Mar-2013	73	212	-56.24%	129.62%	slow/slow
	14-Feb-2020	20-Mar-2020	21-Aug-2020	5	22	-31.81%	47.39%	fast/fast
SX5E	01-Jun-2007	09-Mar-2009	none	92	No recovery	-60.29%	none	slow/no
	14-Feb-2020	20-Mar-2020	26-Mar-2021	5	53	-38.27%	62.11%	fast/moderate

Table 2: Features of the subprimes and COVID-19 drawdown for both the S&P 500 Index and the Euro Stoxx 50 Index. Time is expressed in term of weeks.

Additionally, the table 3 shows that for all periods the distribution of returns are asymmetrical and display important fat tails. Indeed, the estimated skewness are all negative and the estimated kurtosis are significantly higher compare to a normal distribution. Moreover, when combined with the range of the distribution we clearly observed that these periods are subject to important stress, i.e. the maximal loss over one week is much more higher than the maximal gain.

Index	Period		Mean	Std deviation	Max	Min	Skewness	Kurtosis
SPX	31/12/2006	31/12/2011	0.53%	20.74%	11.36%	-20.08%	-0.88	10.59
	31/12/2017	31/12/2022	8.98%	18.32%	11.42%	-16.23%	-1.16	12.17
SX5E	31/12/2006	31/12/2011	-3.45%	24.86%	11.52%	-25.13%	-1.30	11.42
	31/12/2017	31/12/2022	2.13%	20.35%	10.39%	-22.30%	-1.83	17.38

Table 3: Descriptive statistic of the indexes weekly returns. The average and standard deviation are annualized using the a convention of 52 weeks in a year.

Finally for both of these period we use as risk free rates the average returns of the Barclay's benchmark overnight cash index associated to the currency of the S&P 500 and the Euro Stoxx 50[7] (see the table 4 below).

Period		US market	European market
31/12/2006	31/12/2011	2.25%	2.12%
31/12/2017	31/12/2022	1.08%	-0.33%

Table 4: Annualized average risk free rate based on end of week prices of the BXIIBUS0 Index and the BXIIBEU0 Index.

5.1 Simulation based analysis

In this section, we use a non-parametric simulation method to analyze the performance profile and sensitivity to parameters of the previous strategies. In our case, we prefer the use of non-parametric method since they tend to preserve the empirical properties of the considered sample without constraining the process dynamic. Moreover, the periods used for the simulation are very specific in term of distribution.

The considered simulation method is the Circular Block Bootstrap introduced by Politis and Romano [16]. This method is based on a block-resampling mechanism but first wraps the sample around a circle before segmenting it into blocks (for further details on the procedure see appendix A). This approach allows to account for all data with equal proportion and thus provides unbiased estimators, i.e. data at the edge of sample as the same probability of being drawn. Since asset returns exhibits a persistence in their variation the block size plays a crucial role. A too small block size will not preserve this serial dependence while a too wide block size will generate too similar paths. In our application, the block size is determined using the procedure of Politis and White ([17], [18]) which provides an optimal block size based on the sample autocorrelation. We apply this procedure to the absolute returns and not directly to the returns since there are not autocorrelated[8]. The estimated block size obtained for every sample are given in the table 5 below:

Period		SPX	SX5E
31/12/2006	31/12/2011	17	22
31/12/2017	31/12/2022	18	10

Table 5: The estimated block size are expressed in weeks. Additionally for a practical purposes the estimates are rounded toward the nearest integer.

Finally, we consider 5000 simulated sample paths for which we apply the strategies with the baseline parameters to estimate their performance profile. In addition, we change locally the value

[7]The BXIIBUS0 Index and the BXIIBEU0 Index respectively associated to the American and the European economy

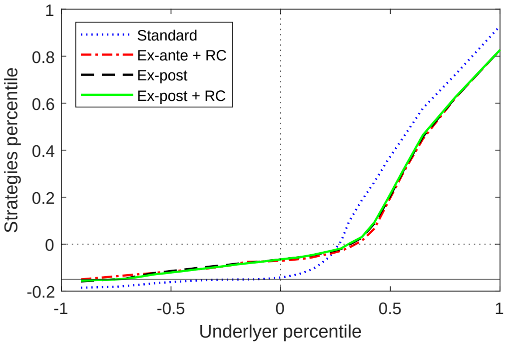
[8]We use the absolute function as a proxy of the returns variability.

of each parameters to analyze the strategy sensitivities. For the sake of simplicity, we limit the sensitivity analysis to the following parameter grid (see table 6):

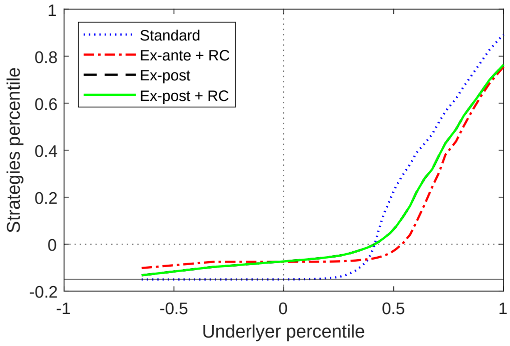
Multiple	**Trigger**	**Threshold**	**Quantile**
m	$\overline{w_{t_k}}$	β	ϵ
3	1%	5%	1%
6	5%	10%	5%
9	15%	15%	10%

Table 6: Parameter grid for the sensitivity analysis. Only one parameter varies at a time while the others are set to the baseline parameters.

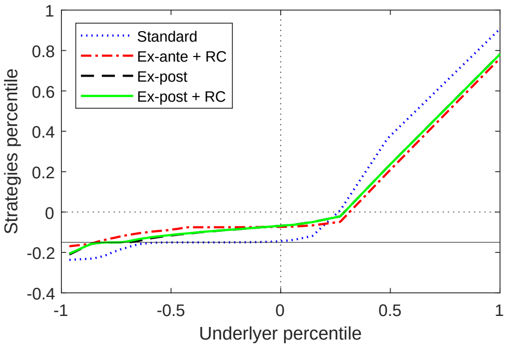
The bootstrap payoff profile (figure 2) of baseline strategies illustrates the risk-reward trade-off between the two margin mechanisms. Ex-post strategies deliver higher performances than ex-ante strategies when the underlyer displays positive returns. On the other hand this extra gain in performance comes at the cost of greater losses when the underlyer price is in a downward configuration. Depending on the period this cost can be either insignificant or detrimental compare to the benefit it provides. For example, from the figure 2b ex-post mechanism provides an important upside participation for a marginal cost. While from the figure 2c we observe the opposite, the upside benefit is marginal compare to the losses increase.



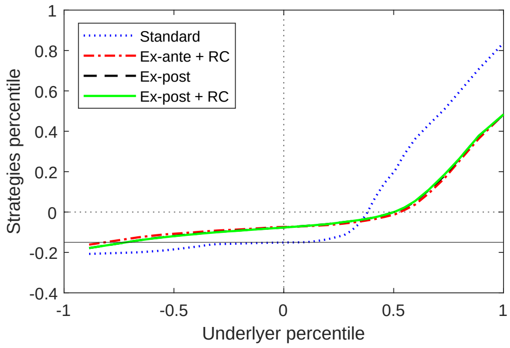
(a) SPX Index from 31/12/2006 to 31/12/2011



(b) SPX Index from 31/12/2016 to 31/12/2022



(c) SX5E Index from 31/12/2006 to 31/12/2011

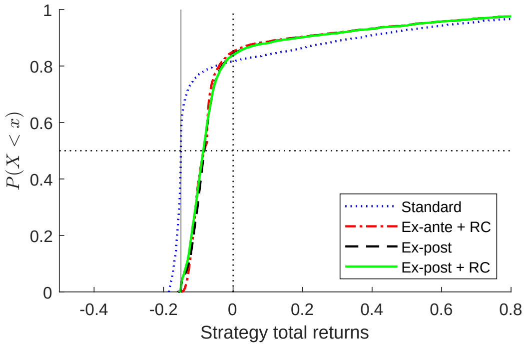


(d) SX5E Index from 31/12/2016 to 31/12/2022

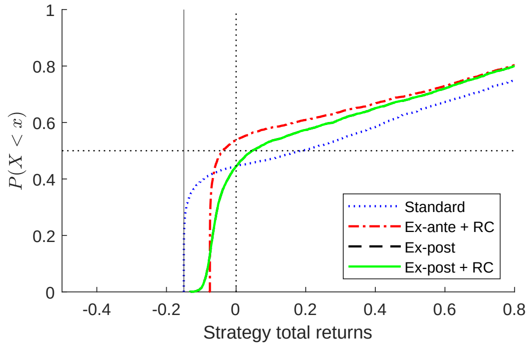
Figure 2: Bootstrap QQ-Plot of the strategies for the baseline parameter set. The dark black line corresponds to the target guarantee. The x-axis and y-axis are capped to 1.

The estimated payoff CDF (figure 3) contributes to this results. Overall the ex-post mechanism is subject to higher losses but also provides an higher probability of having greater returns. As previously, this result is well represented from the figure 3b and 3c. To summarize, ex-post strategies appears more sensitive to the market configuration than the ex-ante strategies.

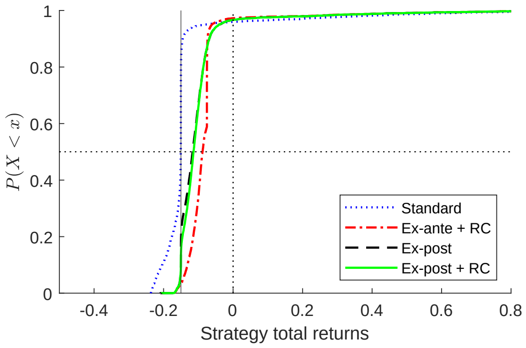
Moreover, the previous figures show the very little impact the risk control has on the payoff profile and distribution. Across all periods there is almost no distinction between the ex-post strategy using the level risk control and the naive strategy. This lack of impact is due to the low occurrences of the risk control application and the aggregation over all the bootstrap samples. As a consequence, the impact of the risk control is strongly dampened.



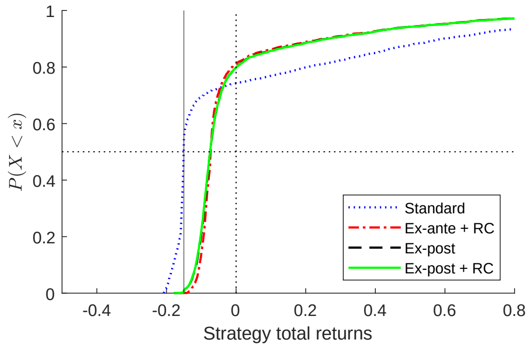
(a) SPX Index from 31/12/2006 to 31/12/2011



(b) SPX Index from 31/12/2016 to 31/12/2022



(c) SX5E Index from 31/12/2006 to 31/12/2011



(d) SX5E Index from 31/12/2016 to 31/12/2022

Figure 3: Bootstrap CDF of the strategies for the baseline parameter set. The dark black line corresponds to the target guarantee.

The sensitivity analysis shows (see tables 7, 8, 9 and 10) interesting and different behaviors between the strategies. First, when focusing on the margin consumption we observe that strategies using the ex-ante triggering mechanism exhibit on average the lowest consumption level and rate[9] over all periods and across all parameters. This result tends to be lessen when considering a higher multiple value ($m = 9$) or a lower quantile level ($\epsilon = 1\%$). Indeed, the multiple level will directly ease the ability to reach the ex-ante triggering event as introduced previously while a reduction of the quantile level increases both the magnitude and the frequency of the floor adjustment.

Additionally for both margin mechanism we note the same the relationship between the parameters and the margin consumption. Independently from the periods, as the parameters become more conservative[10] the use of the margin decreases. For example, when considering the S&P500 case over the period 2017-2022 (see table 9) we note that decreasing the multiple value from 9 to 3 results in not using at all the margin.

Finally there is a direct link between the intensity of the trading activity and the shape of distribution. Indeed, an higher margin consumption yields to increase the volatility and reduce the skewness and the kurtosis. The first observation is straightforward, since the higher the trading activity the higher the exposure into the underlyer. The second observation is implicit the extra activity smooths

[9]The consumption rate corresponds to the number of time a triggering event yields to a floor adjustment. Indeed, the risk control can suggest to increase the margin if the risk is considered too important but since we restrict the floor adjustment to a decrease some cases are ignored.

[10]A decrease of the multiple, the risk control threshold, the ex-post triggering threshold and an increase of the quantile level.

the abrupt change in the returns magnitude as the strategies converge toward their respective floors. It is worth noting that in our results the ex-ante mechanism is less subject to this relation due to its general low margin consumption.

Second, performance wise the results are mixed regarding the periods and the considered performance measures. According to the Sharpe ratio we note that the standard CPPI strategy provides better results overall or in worst cases similar performances compare to the margin based strategies. Within the margin based strategies the results are splitted according to the consider periods. For period with an upward trend ex-post strategies provides better results than the ex-ante strategy. On the other hand, for the period with no recovery ex-post strategies are the worst performer. Such results are not surprising since the ex-post triggering mechanism is by design much more reachable than his ex-ante counterpart. In this context, ex-post strategies benefit from an higher trading activity when the upward trend materializes and conversely suffers from it when only a downward trend is observed. These results must be interpreted cautiously first the standard CPPI used in our analysis starts with a much lower floor making the comparison not entirely homogeneous. Second the difference between the Sharpe ratios are not too significant overall except for specific parameter and period.

However the results obtained using the Omega ratios are consistent across periods and parameters. Margin based strategies provide higher Omega ratios when considering the lowest thresholds, i.e. the risk free rate and 0%. While on the opposite side, for the highest levels, i.e. 5% and 10%[11], the results are inverted. This observation indicates that margin based strategies provides better management of gains over losses when they are relatively small. Conversely, standard CPPI provides a better management for higher losses. This trade-off is due to the design of the margin mechanism. The later increases over time the exposure into the underlying asset making the strategies more vulnerable to several important price drops. Where the standard CPPI has a greater ability to be stuck to its floor and then limits its sensitivity to the underlyer.

Then the comparison between ex-ante and ex-post (with and without a risk control) mechanism shows that the ex-post mechanism provides in all cases better Omega ratios. It is interesting to see that the extra trading activity is profitable in term of gains over losses even if for some periods it leads to an higher global losses (for instance see table 8).

Also note that the impact of the risk control on performances is not clear since we only observe too few effective application of it. Indeed, on average the maximum number of time the risk control is effective is about 5 times across all periods. As previously introduced, the aggregation of the results might flatten the effect on the performances. Indeed from the tables, there are no significant performance differences between the ex-post strategy with and without risk control. However, it is worth noting that the use of the risk control seems to increase the volatility of the strategies. By comparison with the naive strategy this increase indicates that the risk control tends to generate higher floor adjustment and thus a locally[12] greater exposure increase.

[11] The threshold are expressed annually and transformed on a weekly basis using a 52 weeks per year convention.
[12] At the time of the adjustment.

Quantile	1.00%				5.00%				10.00%			
Strategy	**Standard**	**Ex-ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex-ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex-ante + RC**	**Ex-post**	**Ex-post + RC**
Total	-3.23%	-1.27%	-0.99%	-1.18%	-3.23%	-0.66%	-0.99%	-0.99%	-3.23%	-0.70%	-0.99%	-1.01%
Mean (an.)	-0.40%	-0.07%	-0.01%	-0.05%	-0.40%	0.07%	-0.01%	-0.01%	-0.40%	0.06%	-0.01%	-0.01%
Vol. (an.)	12.82%	9.90%	9.97%	10.06%	12.82%	9.67%	9.97%	9.97%	12.82%	9.68%	9.97%	9.98%
Skewness	-2.30	-1.98	-1.87	-1.81	-2.30	-2.27	-1.87	-1.87	-2.30	-2.25	-1.87	-1.87
Kurtosis	25.05	20.50	19.02	18.36	25.05	23.76	19.02	19.01	25.05	23.60	19.02	18.97
SR	-0.29	-0.34	-0.32	-0.32	-0.29	-0.34	-0.32	-0.32	-0.29	-0.34	-0.32	-0.32
$\Omega_{\mathbf{r.f.r.}}$	0.79	0.82	0.85	0.85	0.79	0.79	0.85	0.85	0.79	0.79	0.85	0.85
$\Omega_{0\%}$	0.87	0.91	0.93	0.93	0.87	0.90	0.93	0.93	0.87	0.90	0.93	0.93
$\Omega_{5\%}$	0.68	0.67	0.69	0.70	0.68	0.63	0.69	0.69	0.68	0.63	0.69	0.69
$\Omega_{10\%}$	0.56	0.52	0.53	0.54	0.56	0.49	0.53	0.53	0.56	0.49	0.53	0.53
Margin		5.00%	4.50%	4.20%		6.30%	4.50%	4.50%		6.30%	4.50%	4.50%
# of trig.		15.75	15.70	14.43		1.02	15.70	15.69		1.02	15.70	15.74
no RC		0.00	13.61	8.84		0.84	13.61	13.52		0.84	13.61	13.43
RC		4.08	0.00	2.35		0.08	0.00	0.06		0.08	0.00	0.04
Multiple	**3**				**6**				**9**			
Strategy	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**
Total	-1.88%	1.00%	0.87%	0.87%	-3.23%	-1.27%	-0.99%	-1.18%	-3.81%	-0.89%	-1.47%	-1.29%
Mean (an.)	-0.23%	0.31%	0.28%	0.28%	-0.40%	-0.07%	-0.01%	-0.05%	-0.50%	0.05%	-0.08%	-0.03%
Vol. (an.)	10.18%	7.02%	7.09%	7.09%	12.82%	9.90%	9.97%	10.06%	13.47%	10.43%	11.03%	11.00%
Skewness	-1.74	-1.78	-1.62	-1.62	-2.30	-1.98	-1.87	-1.81	-2.51	-2.71	-2.14	-2.16
Kurtosis	16.97	17.19	15.20	15.20	25.05	20.50	19.02	18.36	29.05	31.47	23.90	24.10
SR	-0.31	-0.33	-0.32	-0.32	-0.29	-0.34	-0.32	-0.32	-0.28	-0.32	-0.30	-0.30
$\Omega_{\mathbf{r.f.r.}}$	0.86	0.85	0.87	0.87	0.79	0.82	0.85	0.85	0.77	0.77	0.84	0.84
$\Omega_{0\%}$	0.93	0.96	0.97	0.97	0.87	0.91	0.93	0.93	0.86	0.89	0.92	0.92
$\Omega_{5\%}$	0.72	0.65	0.68	0.68	0.68	0.67	0.69	0.70	0.67	0.61	0.69	0.69
$\Omega_{10\%}$	0.58	0.48	0.49	0.49	0.56	0.52	0.53	0.54	0.56	0.48	0.55	0.54
Margin		7.50%	6.50%	6.50%		5.00%	4.50%	4.20%		6.10%	3.90%	4.10%
# of trig.		0.00	5.64	5.64		15.75	15.70	14.43		15.01	25.32	29.06
no RC		0.00	5.62	5.62		0.00	13.61	8.84		0.00	17.83	15.68
RC		0.00	0.00	0.00		4.08	0.00	2.35		4.11	0.00	3.91
Threshold	**5%**				**10%**				**15%**			
Strategy	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**
Total	-3.23%	-1.27%	-0.99%	-1.18%	-3.23%	-1.94%	-0.99%	-1.76%	-3.23%	-2.43%	-0.99%	-2.23%
Mean (an.)	-0.40%	-0.07%	-0.01%	-0.05%	-0.40%	-0.22%	-0.01%	-0.18%	-0.40%	-0.33%	-0.01%	-0.29%
Vol. (an.)	12.82%	9.90%	9.97%	10.06%	12.82%	10.18%	9.97%	10.16%	12.82%	10.43%	9.97%	10.32%
Skewness	-2.30	-1.98	-1.87	-1.81	-2.30	-1.78	-1.87	-1.74	-2.30	-1.69	-1.87	-1.67
Kurtosis	25.05	20.50	19.02	18.36	25.05	18.24	19.02	17.62	25.05	17.29	19.02	16.89
SR	-0.29	-0.34	-0.32	-0.32	-0.29	-0.34	-0.32	-0.34	-0.29	-0.34	-0.32	-0.34
$\Omega_{\mathbf{r.f.r.}}$	0.79	0.82	0.85	0.85	0.79	0.83	0.85	0.85	0.79	0.83	0.85	0.84
$\Omega_{0\%}$	0.87	0.91	0.93	0.93	0.87	0.91	0.93	0.92	0.87	0.91	0.93	0.92
$\Omega_{5\%}$	0.68	0.67	0.69	0.70	0.68	0.68	0.69	0.70	0.68	0.68	0.69	0.70
$\Omega_{10\%}$	0.56	0.52	0.53	0.54	0.56	0.54	0.53	0.54	0.56	0.54	0.53	0.55
Margin		5.00%	4.50%	4.20%		3.90%	4.50%	3.60%		3.20%	4.50%	3.20%
# of trig.		15.75	15.70	14.43		15.71	15.70	16.63		15.53	15.70	21.09
no RC		0.00	13.61	8.84		0.00	13.61	7.69		0.00	13.61	6.60
RC		4.08	0.00	2.35		4.19	0.00	1.64		4.31	0.00	1.44
Trigger	**1%**				**5%**				**15%**			
Strategy	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**
Total	-3.23%	-1.27%	-0.37%	-1.02%	-3.23%	-1.27%	-0.99%	-1.18%	-3.23%	-1.27%	-2.06%	-1.81%
Mean (an.)	-0.40%	-0.07%	0.13%	-0.02%	-0.40%	-0.07%	-0.01%	-0.05%	-0.40%	-0.07%	-0.23%	-0.18%
Vol. (an.)	12.82%	9.90%	9.65%	9.75%	12.82%	9.90%	9.97%	10.06%	12.82%	9.90%	11.03%	10.99%
Skewness	-2.30	-1.98	-2.30	-2.14	-2.30	-1.98	-1.87	-1.81	-2.30	-1.98	-1.54	-1.56
Kurtosis	25.05	20.50	24.09	22.27	25.05	20.50	19.02	18.36	25.05	20.50	15.27	15.36
SR	-0.29	-0.34	-0.33	-0.34	-0.29	-0.34	-0.32	-0.32	-0.29	-0.34	-0.31	-0.31
$\Omega_{\mathbf{r.f.r.}}$	0.79	0.82	0.80	0.81	0.79	0.82	0.85	0.85	0.79	0.82	0.86	0.86
$\Omega_{0\%}$	0.87	0.91	0.91	0.91	0.87	0.91	0.93	0.93	0.87	0.91	0.92	0.93
$\Omega_{5\%}$	0.68	0.67	0.63	0.65	0.68	0.67	0.69	0.70	0.68	0.67	0.73	0.73
$\Omega_{10\%}$	0.56	0.52	0.49	0.50	0.56	0.52	0.53	0.54	0.56	0.52	0.59	0.60
Margin		5.00%	6.60%	5.60%		5.00%	4.50%	4.20%		5.00%	2.00%	2.10%
# of trig.		15.75	11.76	5.68		15.75	15.70	14.43		15.75	40.33	43.68
no RC		0.00	11.76	4.76		0.00	13.61	8.84		0.00	9.85	9.43
RC		4.08	0.00	0.92		4.08	0.00	2.35		4.08	0.00	3.03

Table 7: Sentivity analysis for the SPX from 2007 to 2011.

Quantile		**1.00%**				**5.00%**				**10.00%**		
Strategy	**Standard**	**Ex-ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex-ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex-ante + RC**	**Ex-post**	**Ex-post + RC**
Total	-13.77%	-8.07%	-10.02%	-9.81%	-13.77%	-9.75%	-10.02%	-10.54%	-13.77%	-7.79%	-10.02%	-10.02%
Mean (an.)	-2.54%	-1.42%	-1.88%	-1.83%	-2.54%	-1.78%	-1.88%	-1.99%	-2.54%	-1.35%	-1.88%	-1.88%
Vol. (an.)	11.74%	8.66%	8.95%	8.94%	11.74%	9.36%	8.95%	9.24%	11.74%	8.63%	8.95%	8.95%
Skewness	-3.77	-3.89	-3.43	-3.46	-3.77	-3.35	-3.43	-3.22	-3.77	-3.94	-3.43	-3.43
Kurtosis	44.64	45.59	39.30	39.57	44.64	38.86	39.30	36.65	44.64	46.23	39.30	39.30
SR	-0.50	-0.54	-0.58	-0.58	-0.50	-0.53	-0.58	-0.57	-0.50	-0.53	-0.58	-0.58
$\Omega_{\mathbf{r.f.r.}}$	0.65	0.65	0.70	0.70	0.65	0.67	0.70	0.70	0.65	0.64	0.70	0.70
$\Omega_{0\%}$	0.77	0.82	0.82	0.83	0.77	0.82	0.82	0.82	0.77	0.82	0.82	0.82
$\Omega_{5\%}$	0.55	0.50	0.56	0.56	0.55	0.54	0.56	0.57	0.55	0.5	0.56	0.56
$\Omega_{10\%}$	0.44	0.37	0.41	0.41	0.44	0.40	0.41	0.42	0.44	0.36	0.41	0.41
Margin		5.60%	2.70%	2.90%		3.60%	2.70%	2.30%		6.00%	2.70%	2.70%
# of trig.		18.88	36.35	41.63		7.16	36.35	40.60		0.83	36.35	36.35
no RC		0.00	20.74	18.68		0.58	20.74	14.50		0.83	20.74	20.74
RC		5.11	0.00	4.65		2.05	0.00	0.73		0.00	0.00	0.00
Multiple		**3**				**6**				**9**		
Strategy	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**
Total	-11.49%	-5.23%	-6.33%	-6.33%	-13.77%	-8.07%	-10.02%	-9.81%	-14.23%	-12.99%	-11.62%	-12.86%
Mean (an.)	-2.25%	-0.96%	-1.22%	-1.22%	-2.54%	-1.42%	-1.88%	-1.83%	-2.58%	-2.43%	-2.16%	-2.42%
Vol. (an.)	9.21%	6.41%	6.54%	6.54%	11.74%	8.66%	8.95%	8.94%	12.48%	10.84%	10.15%	10.62%
Skewness	-2.98	-3.03	-2.68	-2.68	-3.77	-3.89	-3.43	-3.46	-3.93	-3.55	-3.70	-3.44
Kurtosis	32.50	32.95	28.10	28.10	44.64	45.59	39.30	39.57	47.40	41.35	44.23	40.05
SR	-0.58	-0.58	-0.61	-0.61	-0.50	-0.54	-0.58	-0.58	-0.47	-0.54	-0.55	-0.55
$\Omega_{\mathbf{r.f.r.}}$	0.72	0.71	0.75	0.75	0.65	0.65	0.70	0.70	0.63	0.64	0.68	0.66
$\Omega_{0\%}$	0.82	0.88	0.88	0.88	0.77	0.82	0.82	0.83	0.76	0.77	0.80	0.79
$\Omega_{5\%}$	0.59	0.54	0.59	0.59	0.55	0.50	0.56	0.56	0.54	0.53	0.56	0.55
$\Omega_{10\%}$	0.45	0.38	0.40	0.40	0.44	0.37	0.41	0.41	0.43	0.42	0.42	0.43
Margin		7.50%	4.90%	4.90%		5.60%	2.70%	2.90%		1.50%	2.30%	1.50%
# of trig.		0.00	14.23	14.23		18.88	36.35	41.63		44.20	32.97	41.46
no RC		0.00	13.55	13.55		0.00	20.74	18.68		0.00	23.81	6.74
RC		0.00	0.00	0.00		5.11	0.00	4.65		11.29	0.00	5.60
Threshold		**5%**				**10%**				**15%**		
Strategy	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**
Total	-13.77%	-8.07%	-10.02%	-9.81%	-13.77%	-8.68%	-10.02%	-10.08%	-13.77%	-9.12%	-10.02%	-10.27%
Mean (an.)	-2.54%	-1.42%	-1.88%	-1.83%	-2.54%	-1.56%	-1.88%	-1.89%	-2.54%	-1.66%	-1.88%	-1.93%
Vol. (an.)	11.74%	8.66%	8.95%	8.94%	11.74%	8.73%	8.95%	8.96%	11.74%	8.81%	8.95%	8.99%
Skewness	-3.77	-3.89	-3.43	-3.46	-3.77	-3.77	-3.43	-3.42	-3.77	-3.66	-3.43	-3.39
Kurtosis	44.64	45.59	39.30	39.57	44.64	43.87	39.30	39.20	44.64	42.37	39.30	38.82
SR	-0.50	-0.54	-0.58	-0.58	-0.50	-0.55	-0.58	-0.58	-0.50	-0.56	-0.58	-0.58
$\Omega_{\mathbf{r.f.r.}}$	0.65	0.65	0.70	0.70	0.65	0.66	0.70	0.70	0.65	0.66	0.70	0.70
$\Omega_{0\%}$	0.77	0.82	0.82	0.83	0.77	0.81	0.82	0.82	0.77	0.81	0.82	0.82
$\Omega_{5\%}$	0.55	0.50	0.56	0.56	0.55	0.51	0.56	0.56	0.55	0.52	0.56	0.56
$\Omega_{10\%}$	0.44	0.37	0.41	0.41	0.44	0.38	0.41	0.41	0.44	0.38	0.41	0.41
Margin		5.60%	2.70%	2.90%		4.90%	2.70%	2.60%		4.40%	2.70%	2.50%
# of trig.		18.88	36.35	41.63		18.88	36.35	37.49		18.85	36.35	37.57
no RC		0.00	20.74	18.68		0.00	20.74	17.11		0.00	20.74	15.99
RC		5.11	0.00	4.65		5.15	0.00	3.76		5.17	0.00	3.02
Trigger		**1%**				**5%**				**15%**		
Strategy	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**
Total	-13.77%	-8.07%	-7.77%	-8.12%	-13.77%	-8.07%	-10.02%	-9.81%	-13.77%	-8.07%	-12.20%	-12.11%
Mean (an.)	-2.54%	-1.42%	-1.34%	-1.43%	-2.54%	-1.42%	-1.88%	-1.83%	-2.54%	-1.42%	-2.36%	-2.33%
Vol. (an.)	11.74%	8.66%	8.63%	8.65%	11.74%	8.66%	8.95%	8.94%	11.74%	8.66%	9.83%	9.82%
Skewness	-3.77	-3.89	-3.94	-3.89	-3.77	-3.89	-3.43	-3.46	-3.77	-3.89	-3.03	-3.05
Kurtosis	44.64	45.59	46.16	45.58	44.64	45.59	39.30	39.57	44.64	45.59	33.44	33.58
SR	-0.50	-0.54	-0.53	-0.54	-0.50	-0.54	-0.58	-0.58	-0.50	-0.54	-0.58	-0.58
$\Omega_{\mathbf{r.f.r.}}$	0.65	0.65	0.66	0.66	0.65	0.65	0.70	0.70	0.65	0.65	0.69	0.70
$\Omega_{0\%}$	0.77	0.82	0.82	0.82	0.77	0.82	0.82	0.83	0.77	0.82	0.81	0.81
$\Omega_{5\%}$	0.55	0.50	0.50	0.50	0.55	0.50	0.56	0.56	0.55	0.50	0.58	0.59
$\Omega_{10\%}$	0.44	0.37	0.36	0.37	0.44	0.37	0.41	0.41	0.44	0.37	0.45	0.45
Margin		5.60%	5.80%	5.40%		5.60%	2.70%	2.90%		5.60%	0.80%	0.80%
# of trig.		18.88	22.52	17.92		18.88	36.35	41.63		18.88	76.23	76.69
no RC		0.00	20.61	13.94		0.00	20.74	18.68		0.00	11.22	11.44
RC		5.11	0.00	2.08		5.11	0.00	4.65		5.11	0.00	2.75

Table 8: Sentivity analysis for the SX5E from 2007 to 2011.

Quantile	**1.00%**				**5.00%**				**10.00%**			
Strategy	**Standard**	**Ex-ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex-ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex-ante + RC**	**Ex-post**	**Ex-post + RC**
Total	39.53%	32.90%	35.83%	35.83%	39.53%	32.90%	35.83%	35.83%	39.53%	32.90%	35.83%	35.83%
Mean (an.)	5.41%	4.52%	5.03%	5.03%	5.41%	4.52%	5.03%	5.03%	5.41%	4.52%	5.03%	5.03%
Vol. (an.)	14.07%	11.69%	12.29%	12.29%	14.07%	11.69%	12.29%	12.29%	14.07%	11.69%	12.29%	12.29%
Skewness	-2.09	-2.65	-1.94	-1.94	-2.09	-2.65	-1.94	-1.94	-2.09	-2.65	-1.94	-1.94
Kurtosis	21.64	30.50	19.41	19.41	21.64	30.50	19.41	19.41	21.64	30.50	19.41	19.41
SR	0.19	0.12	0.19	0.19	0.19	0.12	0.19	0.19	0.19	0.12	0.19	0.19
$\Omega_{\mathbf{r.f.r.}}$	1.05	1.01	1.09	1.09	1.05	1.01	1.09	1.09	1.05	1.01	1.09	1.09
$\Omega_{0\%}$	1.11	1.09	1.15	1.15	1.11	1.09	1.15	1.15	1.11	1.09	1.15	1.15
$\Omega_{5\%}$	0.91	0.84	0.93	0.93	0.91	0.84	0.93	0.93	0.91	0.84	0.93	0.93
$\Omega_{10\%}$	0.78	0.69	0.77	0.77	0.78	0.69	0.77	0.77	0.78	0.69	0.77	0.77
Margin		7.50%	6.70%	6.70%		7.50%	6.70%	6.70%		7.50%	6.70%	6.70%
# of trig.		0.00	3.07	3.07		0.00	3.07	3.07		0.00	3.07	3.07
no RC		0.00	3.07	3.07		0.00	3.07	3.07		0.00	3.07	3.07
RC		0.00	0.00	0.00		0.00	0.00	0.00		0.00	0.00	0.00
Multiple	**3**				**6**				**9**			
Strategy	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**
Total	38.06%	29.48%	29.56%	29.56%	39.53%	32.90%	35.83%	35.83%	40.17%	34.63%	38.41%	37.33%
Mean (an.)	5.50%	4.36%	4.38%	4.38%	5.41%	4.52%	5.03%	5.03%	5.45%	4.69%	5.37%	5.16%
Vol. (an.)	12.45%	9.40%	9.41%	9.41%	14.07%	11.69%	12.29%	12.29%	14.51%	12.85%	13.29%	13.25%
Skewness	-1.32	-1.45	-1.43	-1.43	-2.09	-2.65	-1.94	-1.94	-2.18	-2.21	-1.81	-1.88
Kurtosis	10.83	11.86	11.69	11.69	21.64	30.50	19.41	19.41	23.32	22.82	18.03	18.80
SR	0.27	0.22	0.23	0.23	0.19	0.12	0.19	0.19	0.18	0.12	0.21	0.18
$\Omega_{\mathbf{r.f.r.}}$	1.13	1.11	1.11	1.11	1.05	1.01	1.09	1.09	1.02	1.02	1.09	1.08
$\Omega_{0\%}$	1.17	1.18	1.18	1.18	1.11	1.09	1.15	1.15	1.09	1.09	1.15	1.13
$\Omega_{5\%}$	0.98	0.92	0.92	0.92	0.91	0.84	0.93	0.93	0.9	0.87	0.95	0.93
$\Omega_{10\%}$	0.83	0.72	0.72	0.72	0.78	0.69	0.77	0.77	0.78	0.73	0.79	0.78
Margin		7.50%	7.40%	7.40%		7.50%	6.70%	6.70%		4.80%	5.60%	5.00%
# of trig.		0.00	0.28	0.28		0.00	3.07	3.07		21.66	5.80	12.38
no RC		0.00	0.28	0.28		0.00	3.07	3.07		0.00	5.39	3.71
RC		0.00	0.00	0.00		0.00	0.00	0.00		2.14	0.00	1.36
Threshold	**5%**				**10%**				**15%**			
Strategy	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**
Total	39.53%	32.90%	35.83%	35.83%	39.53%	32.90%	35.83%	35.83%	39.53%	32.90%	35.83%	35.83%
Mean (an.)	5.41%	4.52%	5.03%	5.03%	5.41%	4.52%	5.03%	5.03%	5.41%	4.52%	5.03%	5.03%
Vol. (an.)	14.07%	11.69%	12.29%	12.29%	14.07%	11.69%	12.29%	12.29%	14.07%	11.69%	12.29%	12.29%
Skewness	-2.09	-2.65	-1.94	-1.94	-2.09	-2.65	-1.94	-1.94	-2.09	-2.65	-1.94	-1.94
Kurtosis	21.64	30.50	19.41	19.41	21.64	30.50	19.41	19.41	21.64	30.50	19.41	19.41
SR	0.19	0.12	0.19	0.19	0.19	0.12	0.19	0.19	0.19	0.12	0.19	0.19
$\Omega_{\mathbf{r.f.r.}}$	1.05	1.01	1.09	1.09	1.05	1.01	1.09	1.09	1.05	1.01	1.09	1.09
$\Omega_{0\%}$	1.11	1.09	1.15	1.15	1.11	1.09	1.15	1.15	1.11	1.09	1.15	1.15
$\Omega_{5\%}$	0.91	0.84	0.93	0.93	0.91	0.84	0.93	0.93	0.91	0.84	0.93	0.93
$\Omega_{10\%}$	0.78	0.69	0.77	0.77	0.78	0.69	0.77	0.77	0.78	0.69	0.77	0.77
Margin		7.50%	6.70%	6.70%		7.50%	6.70%	6.70%		7.50%	6.70%	6.70%
# of trig.		0.00	3.07	3.07		0.00	3.07	3.07		0.00	3.07	3.07
no RC		0.00	3.07	3.07		0.00	3.07	3.07		0.00	3.07	3.07
RC		0.00	0.00	0.00		0.00	0.00	0.00		0.00	0.00	0.00
Trigger	**1%**				**5%**				**15%**			
Strategy	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**
Total	39.53%	32.90%	33.32%	33.32%	39.53%	32.90%	35.83%	35.83%	39.53%	32.90%	40.79%	40.79%
Mean (an.)	5.41%	4.52%	4.60%	4.60%	5.41%	4.52%	5.03%	5.03%	5.41%	4.52%	5.83%	5.83%
Vol. (an.)	14.07%	11.69%	11.75%	11.75%	14.07%	11.69%	12.29%	12.29%	14.07%	11.69%	13.58%	13.58%
Skewness	-2.09	-2.65	-2.53	-2.53	-2.09	-2.65	-1.94	-1.94	-2.09	-2.65	-1.40	-1.40
Kurtosis	21.64	30.50	28.52	28.52	21.64	30.50	19.41	19.41	21.64	30.50	12.47	12.47
SR	0.19	0.12	0.13	0.13	0.19	0.12	0.19	0.19	0.19	0.12	0.27	0.27
$\Omega_{\mathbf{r.f.r.}}$	1.05	1.01	1.04	1.04	1.05	1.01	1.09	1.09	1.05	1.01	1.13	1.13
$\Omega_{0\%}$	1.11	1.09	1.11	1.11	1.11	1.09	1.15	1.15	1.11	1.09	1.17	1.17
$\Omega_{5\%}$	0.91	0.84	0.86	0.86	0.91	0.84	0.93	0.93	0.91	0.84	1	1
$\Omega_{10\%}$	0.78	0.69	0.70	0.70	0.78	0.69	0.77	0.77	0.78	0.69	0.85	0.85
Margin		7.50%	7.40%	7.40%		7.50%	6.70%	6.70%		7.50%	4.90%	4.90%
# of trig.		0.00	2.32	2.32		0.00	3.07	3.07		0.00	10.11	10.11
no RC		0.00	2.32	2.32		0.00	3.07	3.07		0.00	3.68	3.68
RC		0.00	0.00	0.00		0.00	0.00	0.00		0.00	0.00	0.00

Table 9: Sentivity analysis for the SPX from 2017 to 2022.

Quantile	**1.00%**				**5.00%**				**10.00%**			
Strategy	**Standard**	**Ex-ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex-ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex-ante + RC**	**Ex-post**	**Ex-post + RC**
Total	3.70%	1.22%	1.19%	1.19%	3.70%	1.22%	1.19%	1.19%	3.70%	2.60%	1.19%	2.38%
Mean (an.)	0.52%	0.19%	0.18%	0.18%	0.52%	0.19%	0.18%	0.18%	0.52%	0.44%	0.18%	0.40%
Vol. (an.)	10.86%	6.71%	6.92%	6.95%	10.86%	6.71%	6.92%	6.95%	10.86%	8.23%	6.92%	8.20%
Skewness	-3.61	-2.92	-2.53	-2.52	-3.61	-2.92	-2.53	-2.52	-3.61	-2.38	-2.53	-2.19
Kurtosis	45.07	34.53	28.64	28.53	45.07	34.53	28.64	28.53	45.07	28.86	28.64	25.57
SR	-0.09	-0.11	-0.10	-0.10	-0.09	-0.11	-0.10	-0.10	-0.09	-0.06	-0.10	-0.06
$\Omega_{\mathbf{r.f.r.}}$	0.88	0.94	0.96	0.96	0.88	0.94	0.96	0.96	0.88	0.96	0.96	0.97
$\Omega_{0\%}$	0.86	0.90	0.93	0.93	0.86	0.90	0.93	0.93	0.86	0.93	0.93	0.95
$\Omega_{5\%}$	0.66	0.56	0.59	0.60	0.66	0.56	0.59	0.60	0.66	0.65	0.59	0.66
$\Omega_{10\%}$	0.53	0.39	0.41	0.42	0.53	0.39	0.41	0.42	0.53	0.48	0.41	0.49
Margin		6.00%	5.10%	5.00%		6.00%	5.10%	5.00%		3.80%	5.10%	3.60%
# of trig.		9.90	11.09	11.64		9.90	11.09	11.64		6.08	11.09	11.29
no RC		0.00	10.40	8.23		0.00	10.40	8.23		0.15	10.40	5.16
RC		2.32	0.00	1.65		2.32	0.00	1.65		1.80	0.00	0.79

Multiple	**3**				**6**				**9**			
Strategy	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**
Total	2.23%	-0.17%	0.06%	0.06%	3.70%	1.22%	1.19%	1.19%	3.66%	-0.93%	1.00%	0.13%
Mean (an.)	0.35%	-0.03%	0.01%	0.01%	0.52%	0.19%	0.18%	0.18%	0.47%	-0.26%	0.11%	-0.06%
Vol. (an.)	8.14%	3.90%	4.10%	4.10%	10.86%	6.71%	6.92%	6.95%	11.35%	7.08%	7.98%	7.95%
Skewness	-2.67	-2.77	-2.17	-2.17	-3.61	-2.92	-2.53	-2.52	-4.13	-4.23	-2.87	-3.00
Kurtosis	26.82	28.04	20.72	20.72	45.07	34.53	28.64	28.53	53.60	51.67	34.34	35.33
SR	-0.06	-0.10	-0.05	-0.05	-0.09	-0.11	-0.10	-0.10	-0.10	-0.25	-0.13	-0.17
$\Omega_{\mathbf{r.f.r.}}$	0.96	0.94	0.98	0.98	0.88	0.94	0.96	0.96	0.84	0.79	0.93	0.91
$\Omega_{0\%}$	0.93	0.89	0.94	0.94	0.86	0.90	0.93	0.93	0.82	0.75	0.90	0.88
$\Omega_{5\%}$	0.67	0.45	0.50	0.50	0.66	0.56	0.59	0.60	0.63	0.47	0.60	0.58
$\Omega_{10\%}$	0.51	0.27	0.29	0.29	0.53	0.39	0.41	0.42	0.51	0.34	0.43	0.42
Margin		7.50%	6.50%	6.50%		6.00%	5.10%	5.00%		4.50%	4.10%	3.50%
# of trig.		0.00	5.11	5.11		9.90	11.09	11.64		12.76	18.60	28.71
no RC		0.00	5.07	5.07		0.00	10.40	8.23		0.00	15.16	12.78
RC		0.00	0.00	0.00		2.32	0.00	1.65		4.12	0.00	2.97

Threshold	**5%**				**10%**				**15%**			
Strategy	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**
Total	3.70%	1.22%	1.19%	1.19%	3.70%	1.71%	1.19%	1.57%	3.70%	2.17%	1.19%	1.98%
Mean (an.)	0.52%	0.19%	0.18%	0.18%	0.52%	0.28%	0.18%	0.25%	0.52%	0.36%	0.18%	0.33%
Vol. (an.)	10.86%	6.71%	6.92%	6.95%	10.86%	7.24%	6.92%	7.32%	10.86%	7.70%	6.92%	7.71%
Skewness	-3.61	-2.92	-2.53	-2.52	-3.61	-2.51	-2.53	-2.32	-3.61	-2.33	-2.53	-2.18
Kurtosis	45.07	34.53	28.64	28.53	45.07	28.99	28.64	25.99	45.07	27.00	28.64	24.48
SR	-0.09	-0.11	-0.10	-0.10	-0.09	-0.09	-0.10	-0.08	-0.09	-0.07	-0.10	-0.07
$\Omega_{\mathbf{r.f.r.}}$	0.88	0.94	0.96	0.96	0.88	0.95	0.96	0.97	0.88	0.96	0.96	0.97
$\Omega_{0\%}$	0.86	0.90	0.93	0.93	0.86	0.92	0.93	0.94	0.86	0.93	0.93	0.95
$\Omega_{5\%}$	0.66	0.56	0.59	0.60	0.66	0.61	0.59	0.63	0.66	0.64	0.59	0.65
$\Omega_{10\%}$	0.53	0.39	0.41	0.42	0.53	0.44	0.41	0.44	0.53	0.46	0.41	0.47
Margin		6.00%	5.10%	5.00%		5.20%	5.10%	4.60%		4.50%	5.10%	4.10%
# of trig.		9.90	11.09	11.64		9.86	11.09	9.11		9.78	11.09	9.63
no RC		0.00	10.40	8.23		0.00	10.40	6.35		0.00	10.40	5.49
RC		2.32	0.00	1.65		2.33	0.00	1.14		2.36	0.00	1.07

Trigger	**1%**				**5%**				**15%**			
Strategy	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**
Total	3.70%	1.22%	0.75%	1.15%	3.70%	1.22%	1.19%	1.19%	3.70%	1.22%	1.92%	1.46%
Mean (an.)	0.52%	0.19%	0.10%	0.18%	0.52%	0.19%	0.18%	0.18%	0.52%	0.19%	0.31%	0.23%
Vol. (an.)	10.86%	6.71%	6.28%	6.65%	10.86%	6.71%	6.92%	6.95%	10.86%	6.71%	8.77%	8.58%
Skewness	-3.61	-2.92	-3.60	-2.96	-3.61	-2.92	-2.53	-2.52	-3.61	-2.92	-2.24	-2.35
Kurtosis	45.07	34.53	45.27	35.25	45.07	34.53	28.64	28.53	45.07	34.53	23.96	25.12
SR	-0.09	-0.11	-0.15	-0.12	-0.09	-0.11	-0.10	-0.10	-0.09	-0.11	-0.07	-0.09
$\Omega_{\mathbf{r.f.r.}}$	0.88	0.94	0.90	0.94	0.88	0.94	0.96	0.96	0.88	0.94	0.96	0.95
$\Omega_{0\%}$	0.86	0.90	0.85	0.90	0.86	0.90	0.93	0.93	0.86	0.90	0.94	0.93
$\Omega_{5\%}$	0.66	0.56	0.50	0.55	0.66	0.56	0.59	0.60	0.66	0.56	0.70	0.68
$\Omega_{10\%}$	0.53	0.39	0.35	0.39	0.53	0.39	0.41	0.42	0.53	0.39	0.53	0.52
Margin		6.00%	6.70%	6.00%		6.00%	5.10%	5.00%		6.00%	2.40%	2.40%
# of trig.		9.90	8.35	3.19		9.90	11.09	11.64		9.90	30.98	38.67
no RC		0.00	8.35	2.23		0.00	10.40	8.23		0.00	9.57	10.06
RC		2.32	0.00	0.96		2.32	0.00	1.65		2.32	0.00	1.59

Table 10: Sentivity analysis for the SX5E from 2017 to 2022.

5.2 Empirical based analysis

In this part, we illustrate the baseline strategies using the real market data. The objective is not to determine which strategies provide the best performances but to highlight the advantage and inconvenient of each margin mechanisms. Indeed, since we consider very specific periods we cannot draw general conclusions without a large bias.

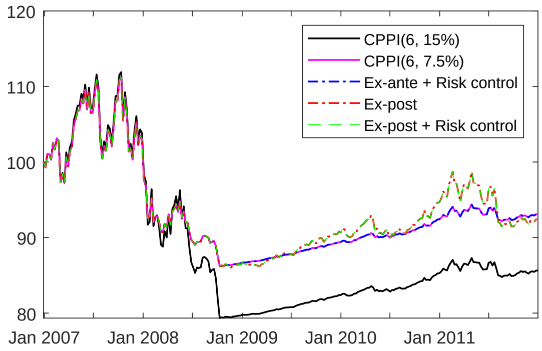
The analysis of the strategies cumulative price index and exposure (figures 4 and 5) shows that the ex-post triggering mechanism provides an effective approach to preserve an exposure independently from the market configuration. On the other hand, the ex-ante triggering mechanism depends on an implicit risk environment determined by the strategy parameters. As previously introduced reaching this environment requires very strong conditions not always easy to fulfill. Therefore, for the S&P 500 case over the period 2007-2011 the ex-ante strategy almost get monetized with an exposition close to 0% (figure 5b) and misses out a significant part of the recovery (figure 4a). Despite being a period of historical high stress the current market environment was not considered bad enough to trigger a floor adjustment (see the performance table 11).

However, for the periods associated to the Euro Stoxx 50 where the risk control is effective the floor decreases significantly. From the performance table 11 we observe that the margin of the strategy using the risk control has been almost entirely consumed for the period 2007-2011 and completely for the period 2017-2022. In case of a non risk control event the margin is only used to regain up to a 5% exposure which do not require an important amount of the margin. We deduce that the risk control by itself leads to an important margin consumption. In the specific case of the Euro Stoxx 50 over the period 2017-2022, such adjustment level when combined to the ex-post triggering mechanism allows to compensate its low exposure increase.

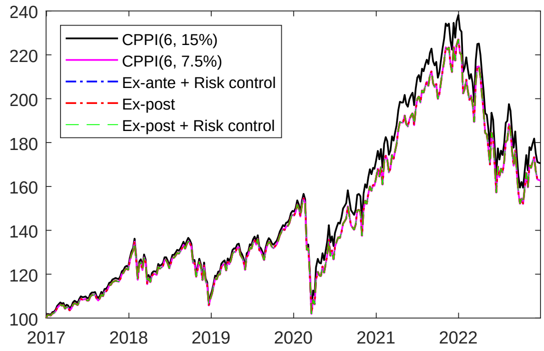
When focusing on the performance, the regain in exposure not always end up in higher gains since it only increase the sensitivity of the strategy to the underlyer. As a consequence, the only way to benefit from this additional exposure is that an emerging trend last until the holding period. Otherwise, an higher margin consumption leads to greater losses as illustrated in the table 11 for both index over the period 2007-2011. Graphically, we observe that the margin mechanism allows to capture an early upward trend but every time this trend reverse all the accumulated gains are lost. Moreover, a part of the margin has been consumed and with it a part of its ability to face other drawdowns.

The analysis of the performance ratio balances the previous result. Indeed, for the Euro Stoxx 50 case over the period 2017-2022 even if a large part of accumulated gains are lost at the end of the period the entire consumption of the margin provides a significant increase of all performance ratios. Similar results are observed for this index over the period 2007-2011, in this scenario all margin strategies provides better Omega ratio compare to the standard CPPI. Regardless the inability of the margin based strategies to preserve their accumulated gains this mechanism tends to improve the management of the gains over the losses.

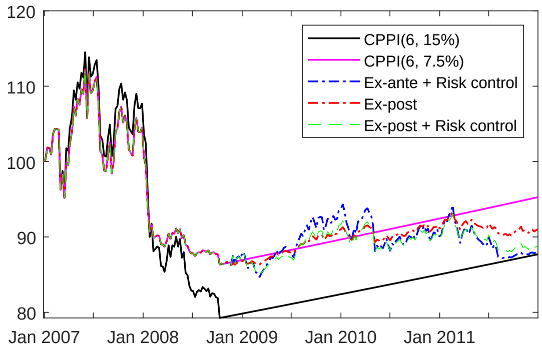
To summarize margin based strategies provides an efficient approach to preserve a certain level of exposure and thus benefit from upside market configuration. The ex-ante mechanism suffers from its dependence to the risk control activation condition. However when the risk control is activated it results in a significant increase of the exposure. Conversely, the ex-post triggering mechanism deliver a consistent regain in exposure independently from the risk environment but fails to provide a rapid increase of the exposure. The combination of the ex-post mechanism and the risk control mitigates both drawbacks while preserving the benefit.



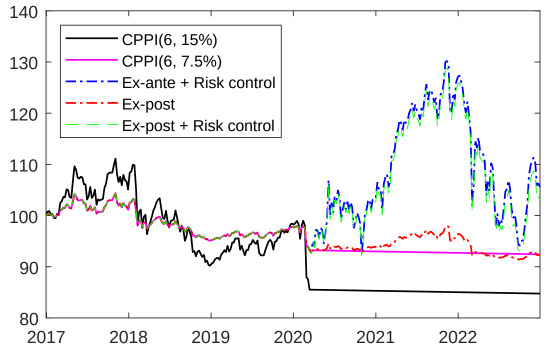
(a) SPX Index from 31/12/2006 to 31/12/2011



(b) SPX Index from 31/12/2016 to 31/12/2022

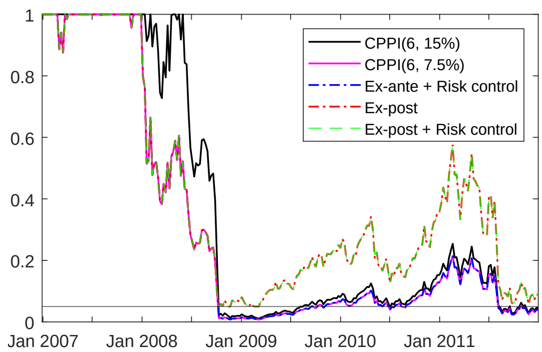


(c) SX5E Index from 31/12/2006 to 31/12/2011

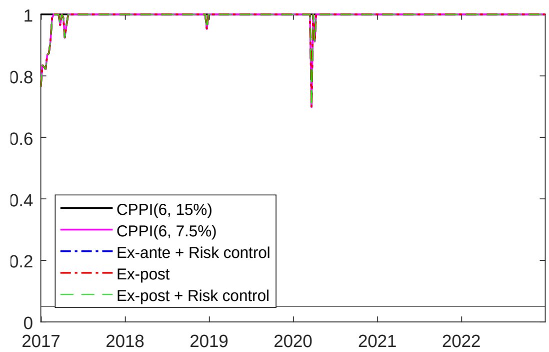


(d) SX5E Index from 31/12/2016 to 31/12/2022

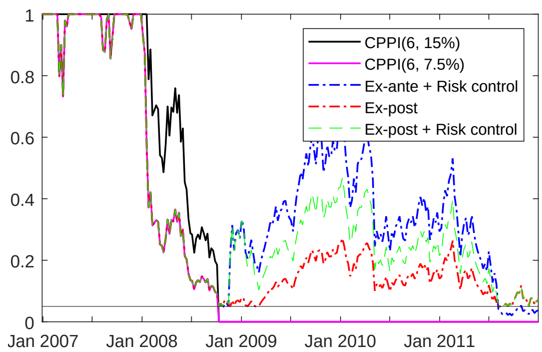
Figure 4: Strategies cumulative price index.



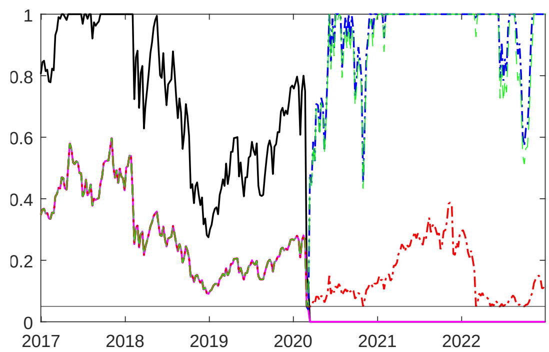
(a) SPX Index from 31/12/2006 to 31/12/2011



(b) SPX Index from 31/12/2016 to 31/12/2022



(c) SX5E Index from 31/12/2006 to 31/12/2011



(d) SX5E Index from 31/12/2016 to 31/12/2022

Figure 5: Strategies relative exposure into the underlyer. The dark black line corresponds to the ex-post triggering threshold, i.e. 5%.

Index	S&P 500							
Period	**2007-2011**				**2017-2022**			
Strategy	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex-ante + RC**	**Ex-post**	**Ex-post + RC**
Total	-14.36%	-6.90%	-7.60%	-7.60%	70.56%	62.66%	62.66%	62.66%
Mean (an.)	-2.55%	-1.04%	-1.13%	-1.13%	11.48%	10.64%	10.64%	10.64%
Vol. (an.)	10.31%	8.75%	9.35%	9.35%	22.70%	22.46%	22.46%	22.46%
Skewness	-1.06	-1.44	-1.21	-1.21	-0.63	-0.73	-0.73	-0.73
Kurtosis	8.00	10.32	8.11	8.11	6.49	6.42	6.42	6.42
SR	-0.47	-0.38	-0.36	-0.36	0.46	0.43	0.43	0.43
$\Omega_{\mathbf{r.f.r.}}$	0.78	0.82	0.85	0.85	1.20	1.18	1.18	1.18
$\Omega_{0\%}$	0.88	0.94	0.95	0.95	1.22	1.20	1.20	1.20
$\Omega_{5\%}$	0.68	0.69	0.74	0.74	1.12	1.10	1.10	1.10
$\Omega_{10\%}$	0.54	0.52	0.58	0.58	1.03	1.01	1.01	1.01
Margin		7.50%	6.10%	6.10%		7.50%	7.50%	7.50%
# of trig.		0	7	0		0	0	0
# non RC		0	7	7		0	0	0
# RC		0	0	7		0	0	0

Index	Euro Stoxx 50							
Period	**2007-2011**				**2017-2022**			
Strategy	**Standard**	**Ex ante + RC**	**Ex-post**	**Ex-post + RC**	**Standard**	**Ex-ante + RC**	**Ex-post**	**Ex-post + RC**
Total	-12.30%	-11.97%	-8.99%	-11.17%	-15.24%	5.55%	-7.77%	3.03%
Mean (an.)	-2.04%	-2.00%	-1.42%	-1.86%	-2.37%	1.89%	-1.26%	1.45%
Vol. (an.)	10.63%	10.38%	9.50%	9.93%	8.52%	13.99%	4.03%	13.76%
Skewness	-1.07	-1.01	-1.25	-1.10	-2.98	-0.77	-2.04	-0.84
Kurtosis	11.87	9.50	13.17	11.14	24.66	11.22	12.42	11.64
SR	-0.39	-0.40	-0.37	-0.40	-0.24	0.16	-0.24	0.13
$\Omega_{\mathbf{r.f.r.}}$	0.78	0.84	0.82	0.83	0.87	1.07	0.90	1.06
$\Omega_{0\%}$	0.89	0.92	0.92	0.92	0.85	1.06	0.87	1.05
$\Omega_{5\%}$	0.66	0.74	0.70	0.73	0.63	0.90	0.50	0.89
$\Omega_{10\%}$	0.51	0.60	0.53	0.57	0.47	0.77	0.28	0.75
Margin		2.50%	4.90%	2.80%		0.00%	5.60%	0.00%
# of trig.		99	12	8		1	13	3
# non RC		1	12	7		0	13	2
# RC		6	0	1		1	0	1

Table 11: Empirical performance table of the strategies applied to both index over the two periods

6 Conclusion

In this paper, we introduce a new version of the CPPI strategy by using a margin based dynamic to target a minimum market exposure. Our approach consists in developing conditional floors directly driven by the strategy exposure into the underlying asset. The floor is adjusted downward through the margin consumption to maintain the required exposure. Also we use a quantile based method to control the associated gap-risk of these strategies. The objective of this control is to take over the floor adjustment when the gap-risk is too important.

Our numerical analysis suggests that the floor process allows to reduce the cash-lock risk while ensuring a better participation in potential market increases. The combination of both minimum exposure constraints and risk controls provides, respectively, a consistent and market-risk adapted exposure mechanism. Performance wise the key improvement results in a stable increase of the Omega ratio over all the considered scenarios.

Some complementary works would, to consider rebuilding the margin over time based on the strategy past performances or considering alternative instruments to maintain or regain exposure into the market such as options.

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A Circular block bootstrap illustration

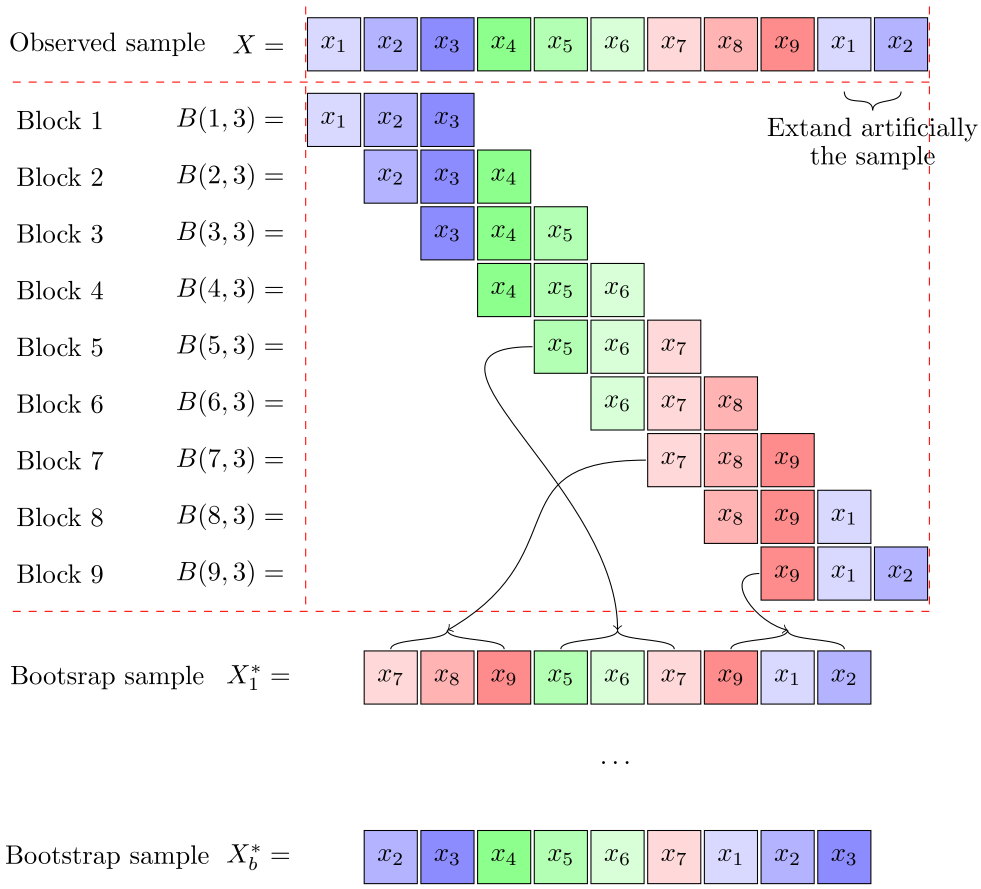

Figure 6: Representation of the circular block Bootstrap for a sample of 9 elements and a block size of 3 elements.