On the Diversification and Rebalancing Returns: Performance Comparison of Constant Mix versus Buy-and-Hold Strategies

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Abstract

This paper examines and compares the constant mix and buy-and-hold portfolio strategies. To this end, we examine and illustrate their performances using various criteria such as comparison of their payoffs, basic properties of their return cumulative distribution functions and their performances with respect to Kappa measures. We also introduce the notion of compensating variation to gauge their respective expected utilities. Our study reveals that, even if the constant mix payoff is more often higher than the buy-and-hold payoff (the probability is around 66%, at least in the GBM framework), this superiority is not very significant. Indeed, for example, when the constant mix is preferable, the compensating variation of the buy-and-hold is weak, whereas, when the buy-and-hold, the compensating variation of the constant mix can be very high. Therefore, when buy-and-hold strategy outperforms rebalancing one with respect to an utility function, it is far more significantly. These results are confirmed by the empirical study that we conduct.

JEL classification: G11, C58, G41.

Keywords: Diversification return; Constant mix portfolio; Buy-and-hold strategy; Compensating variation.

1 Introduction

There is an abundant literature on the comparative performances of constant mix (also called rebalancing) and buy-and-hold portfolio strategies. However, as pointed out by Cuthbertson et al (2015), facing the multitude of results established in various contexts (time periods, financial parameter values, rebalancing rules...), it is difficult to synthesize the literature in order to reach a definitive conclusion on this issue. Buetow et al (2002) investigate the effects of various rebalancing strategies on the risk and return of a multi-asset class portfolio. They argue that rebalancing the portfolio can improve net cost performance while controlling risk. Taking account of transaction costs, Donohue and Yip (2003) manage to identify tractable conditions that determine the no-trade region, which enhances the effectiveness of optimal rebalancing. Using a 30-year financial market data set of the United States, the United Kingdom and Germany, Dichtl et al (2016) analyze both strategies on stock-bond portfolios. They conclude that, even if the portfolio weight of stocks is very low, a frequent rebalancing significantly enhances risk-adjusted portfolio performance for all analyzed countries and all risk-adjusted performance measures. Tsai (2001) considers five stock-bond portfolios with a 20%, 40%, 60%, 80%, and 98% equity allocation, corresponding to different risk profiles of institutional investors. His conclusions are all in favour of an outperformance of the constant mix strategy whatever the rebalancing strategy. Wise (1996) shows that a rebalancing strategy outperforms a buy-and-hold one about two thirds of the time when the assets have the same expected return, but, when buy-and-hold strategy outperforms rebalancing one, it is far more significantly. Additionally, Wise (1996) suggests that the comparison of both strategies may depend on investor's risk aversion. Quian (2014) provides a statistical comparison of both portfolio values at maturity, especially their expected values and variances, for various assumptions about return dynamics and portfolio weightings. For long-only portfolios, he shows that buy-and-hold strategy induces higher expected value but also higher variance of terminal wealth. However, for long-only portfolios, mean-reverting returns are more favorable for fixed-weight portfolios whereas it is the converse for trending returns.

However, Hallerbach (2014) and Cuthbertson et al (2015) point out that it is necessary to investigate whether the portfolio performances comes from the rebalancing process (in such case, we refer to "rebalancing returns")[1] and/or from diversification effect ("diversification returns")[2]. Indeed, diversification provides a benefit in one period but this diversification benefit may vanish if you do not rebalance. Additionally, Cuthbertson et al (2015) illustrate how the apparent advantages of rebalanced strategies over infinite horizons give an inaccurate impression of their performance over finite horizons.

It is commonly accepted that constant mix strategy reduces risk but incurs transaction costs and may not provide a sufficiently significant performance when market rises. It has often been shown

[1] "Rebalancing return" (also called "rebalancing premium") is usually associated to long-horizon investment with the notions of growth-optimal investing, volatility pumping and the Kelly criterion (see Kelly, 1956).

[2] The well-known term "diversification return" has been introduced by Booth and Fama (1992).

that rebalancing induces higher Sharpe ratios than buy-and-hold (see, for example, Boscaljon et al., 2008). For the long-only portfolios (i.e. the weights lie between 0 and 1 as usually assumed for the standard constant mix strategy), the constant mix strategy is a contrarian strategy, meaning that it increases the position in the risky asset when it rises and decreases its position when it makes money.[3] As such, rebalancing induces a lower expected return and a lower variance. Its Sharpe ratio can be also higher than the buy-and-hold's one (see Wise, 1997; Quian, 2014).[4]Rebalancing strategy is sometimes highlighted as a way to take better advantage of market inefficiencies (see e.g. Sharpe (2010) who disagrees with this approach). Other authors argue that constant mix strategy beats buy-and-hold over the long run, in particular when using the Kelly criterion (see Kelly, 1956).

The aim of this paper is to investigate the comparison of rebalancing (i.e. constant mix) portfolio strategy[5] with the buy-and-hold one in various portfolio optimization frameworks. Our contribution is twofold:

- We begin by detailing their main respective properties. For this purpose, we introduce a financial modelling based on diffusion with jumps and various criteria to gauge their performances. First, we compare their respective payoffs for the one risky asset case. We prove that, for the geometric Brownian motion case (GBM), they always intersect at two points. Between these two values, the constant mix strategy provides a higher payoff. We examine the probability of such event, showing that it usually lies between 40% and 60%, according to financial market parameters and to weight invested in the risky asset. We prove also that the difference between constant mix and buy-and-hold returns, namely the rebalancing return, is maximal at a value of the risky asset that does not depend on its trend. We examine also the impact of jumps. We demonstrate that, if all relative jumps have the same signs, the buy-and-hold return is higher than the rebalancing return. This is due to the contrarian behavior of the rebalancing strategy, as illustrated by its negative Gamma in the GBM case. Second, we compare their first four moments. As a by-product, we extend previous results about the comparison of their expectations and variances, which have been previously established only in the i.i.d. case (see e.g. Cheng and Deets, 1971; Wise, 1996). Indeed, we prove that the expectation of the rebalancing strategy is always smaller than that of the buy-and-hold one, which is also true for their variances. This result is established for a jump-diffusion process where both drift and volatility are deterministic but not necessarily constant. In this framework, we also show that both the skewness and the kurtosis of rebalancing strategy are always smaller than that of the buy-and-hold one. Third, we study their respective cumulative

[3]See Perold and Sharpe (1988).

[4]Considering the period from 1995 to 2004, Harjoto and Jones (2006) shows that rebalancing strategy with an incorporated no-trade interval of 15% induces both the highest average return and the lowest standard deviation which also results in the highest Sharpe ratio. However, using data of main US indices over the time period from 1926 to 2009, Jaconetti et al (2010) show that buy-and-hold exhibits the highest average annualized return with a value of 9.1% after an investment period of 84 years, but also the highest volatility with a value of 14.4% due to an average stock allocation of 84.1%.

[5]We only consider the standard constant mix strategy, namely the long only position corresponding to a weight on the risky asset lying between 0 and 1 (for other cases with leverage effects with or without portfolio insurance, see Leland, 1980; Black and Perold, 1992; Bertrand and Prigent, 2005, 2013; Quian, 2012).

distribution functions (cdf). In the GBM case, we prove that they intersect always at two points. Then, we can show that none of these two portfolio strategies can dominate the other one at the first-order stochastic dominance or even at the second-order. Fourth, we analyze their performances using not only the Sharpe ratio but also the Kappa measures considered by Kaplan and Knowles (2004). These latter performance measures include the Omega performance introduced by Keating and Shadwick (2002) (see also Bernardo and Ledoit (2000) for a special case) and the Sortino ratio (see Sortino and Price, 1994). As emphasized by for example Boscaljon et al. (2008), we show that rebalancing induces higher Sharpe ratios than buy-and-hold. However, the difference is small. For the Sharpe Omega ratio, it is the converse: rebalancing induces smaller Sharpe Omega ratios than buy-and-hold, due to the impact of loss aversion. Finally, we introduce the notion of compensating variation to compare the two portfolio strategies by using the well-known expected utility criterion. The compensating variation approach has been introduced by Hicks (1939), Nobel prize winner in economics, and by De Palma and Prigent (2008, 2009) in finance. It allows to quantitatively measure the monetary loss of not receiving the best portfolio. Our findings illustrate that, when the constant mix is preferable, the compensating variation of the buy-and-hold is weak, whereas, when the buy-and-hold, the compensating variation of the constant mix can be very high. It means that, when buy-and-hold strategy outperforms rebalancing one with respect to an utility function, it is far more significantly.[6]

- We provide empirical evidence of the robustness of most of previous theoretical results. Indeed, assessment of long term financial investment through dynamic portfolio strategies relies mostly on parametric Monte Carlo simulations of stochastic processes as soon as we relax the geometric Brownian motion assumption to take account of more complex dynamics such as stochastic nature of the volatility (Hull and White, 1987; Heston, 1993, ...). To overcome the drawbacks of any parametric method such as the estimation risk and the uncertainty in the nature of the return generating process, alternatives have been introduced in the academic literature. The so-called statistical bootstrap method is a possible solution. It is a computer-intensive resampling procedure which generates holding period return (possibly long term one) distribution from the sample monthly (or weekly or daily) returns. In such framework, we can better fit actual financial data.

Our empirical analysis is based on monthly data that cover the sample period from January 1950 to July 2019. The US stock market is represented by the S&P 500 Total return index (dividends included). From the 10-year Treasury constant maturity rate time-series, we approximate long term bond total return using the usual loglinear approximation formula described in chapter 10 of Campbell, Lo and MacKinlay (1997). The 1-month TBill return is from Ibbotson and Associates, Inc. For all portfolio weights (from 1% to 99% in the SP 500), the mean return of the rebalancing strategy is below the buy-and-hold mean return. The same property is true for the volatility. We note also that the probability that the rebalancing strategy ends up with a higher portfolio value than the buy-and-hold strategy is significantly low, in particular for long time horizons. We also compute a Sharpe ratio like measure which is simply the ratio of the mean return and of the volatility

[6] This has been noted by Wise (1996) but only when comparing the payoffs.

of the strategy return. This statistics is clearly in favor of the CM strategy. Considering higher moments of strategy returns (i.e. skewness and kurtosis), the buy-and-hold strategy exhibits much higher level of skewness and kurtosis than the rebalancing strategy. The Sharpe ratio is in favor of the rebalancing strategy. We show also that most of the time the buy-and-hold strategy dominates the rebalancing strategy in the compensating variation sense. Therefore, all these results are in accordance with those established for the theoretical model.

The paper proceeds as follows. Section 2 describes the financial modelling of our theoretical study. Section 3 provides the main theoretical results about comparison of constant mix and buy-and-hold strategies in our general setting, using main performance criteria. Section 4 examines the empirical application. Finally, Section 5 concludes. Most of the proofs are gathered in Appendix which provides also several complementary results.

2 The financial market and the portfolio strategies

In what follows, we consider d financial assets S_i driven by a $d-$dimensional standard Brownian motion $W_t = (W_{i,t})_{1\leq i\leq d}$ and a multivariate point process $N_t = (N_{i,t})_{1\leq i\leq d}$:

$$dS_{i,t} = S_{i,t}(\mu_i(t, S_{i,t})dt + \sigma_i(t, S_{i,t})dW_{i,t} + \delta_i(t, S_{i,t})dN_{i,t}), \tag{1}$$

where Σ and Σ_c are respectively the volatility and correlation matrices given by:

$$\Sigma = \begin{bmatrix} \sigma_1(,) & 0 & 0 & 0 \\ 0 & \sigma_2(,) & 0 & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & 0 & \sigma_d(,) \end{bmatrix}, \ \Sigma_c = \begin{bmatrix} 1 & \rho_{1,2} & ... & \rho_{1,d} \\ \rho_{1,2} & 1 & ... & ... \\ ... & ... & ... & \rho_{d-1,d} \\ \rho_{1,d} & ... & \rho_{d-1,d} & 1 \end{bmatrix}.$$

Functions $\mu(t,s) = (\mu_i(t,s_i))_{1\leq i\leq d}$ and $\sigma(t,s) = (\sigma_i(t,s_i))_{1\leq i\leq d}$ are defined on $[0,T] \times \mathbb{R}^{d+}$ with values in $\mathbb{R}^d$ (with, for all i, $\sigma_i(t,s) > 0$). They satisfy the usual conditions to guarantee the existence, uniqueness and positivity of the solution of this stochastic differential equation (see Jacod and Shiryaev, 2003). Note that the relative jump of the risky asset S_i at any jump time $T_{i,n}$ is given by:

$$\frac{\Delta S_{i,T_{i,n}}}{S_{i,T_{i,n}-}} = \delta_i(T_{i,n}, S_{i,T_{i,n}-}).$$

Recall that, when μ, σ and δ are constant and N is a Poisson process, each process S_i is an exponential of a process with independent and stationary increments, also called Lévy process (see e.g. Merton, 1976, and for detailed explanations of dynamics with jumps, see Last and Brandt, 1995; Prigent, 2001; Cont and Tankov, 2004).

When μ and σ are constant and $\delta(t,s) = 0$, we recover the multidimensional geometric Brownian motion. In that case, the variance-covariance matrix Σ_S of asset prices S is given by: $\Sigma_S =$

$\left(\sigma_{S_i,S_j}\right)_{1\leq i,j\leq d}$ with

$$\sigma_{S_i,S_j} = S_{i,0}S_{j,0}\exp\left[\left(\mu_i+\mu_j\right)t\right]\left(\exp\left[\left(\sigma_i\sigma_j\rho_{i,j}\right)t\right]-1\right).$$

Since the risky asset prices are defined from the relations $dS_{i,t} = S_{i,t}(\mu_i dt + \sigma_i dW_{i,t})$, we deduce that:

$$S_{i,t} = S_{i,0}\exp((\mu_i - 1/2\sigma_i^2)t + \sigma_i W_{i,t}).$$

Therefore, we get:

$$W_{i,t} = Log\left[S_{i,t}/S_{i,0}\right] - \left(\frac{\mu_i}{\sigma_i} - 1/2\sigma_i)t\right),$$

which allows to compute the portfolio values as functions of the risky assets, as illustrated in Relation (5). Diffusion processes with stochastic volatilities can be introduced as well but, by varying the volatility level σ for also different values of the drift μ, main features of portfolio strategies with respect to volatility can be illustrated.

Using previous financial modelling, we deduce the buy-and-hold return $R_T^{BH} = \frac{V_T^{BH}}{V_0}$. For the one risky asset case, at time 0, the investor chooses to invest the weight w on the risky asset and $(1-w)$ on the risk free asset with return e^{rT}. Thus we have:

$$R_T^{BH} = (1-w)e^{rT} + w\frac{S_T}{S_0},$$

from which we deduce:

$$R_T^{BH} = (1-w)e^{rT} + w\exp\left[\int_0^T\left[\mu(t,S_t) - \frac{1}{2}\sigma(t,S_t)^2\right]dt + \int_0^T\sigma_t dW_t\right]\prod_{T_n\leq T}\left(1+\delta(T_n,S_{T_n-})\right). \quad (2)$$

Therefore, the cumulative jumps are equal to $w\prod_{T_n\leq T}(1+\delta(T_n,S_{T_n-}))$.

For the constant mix strategy, first consider the one risky asset case: at any time during the management period $[0,T]$, the investor chooses to invest the fixed weight w on the risky asset and $(1-w)$ on the risk free asset with return e^{rT}. Thus, we deduce that the constant mix portfolio value must satisfy:

$$\frac{dV_t^{CM}}{V_{t-}} = (1-w)rdt + w\frac{dS_t}{S_{t-}} = [(1-w)rdt + w\mu(t,S_t)]dt + w\sigma(t,S_t)dW_t + w\delta(t,S_t)dN_t,$$

from which, using Ito's formula, we deduce:

$$R_T^{CM} = \frac{V_T^{CM}}{V_0} = \exp\left[\int_0^T\left[(1-w)r + w\mu(t,S_t) - \frac{1}{2}w^2\sigma_t^2\right]dt + w\int_0^T\sigma_t dW_t\right]\prod_{T_n\leq T}\left(1+w\delta(T_n,S_{T_n-})\right). \quad (3)$$

Note that the relative jumps of the constant mix portfolio at any jump time T_n are given by:

$$\frac{\Delta V_{T_n}^{CM}}{V_{T_n-}^{CW}} = w\delta(T_n, S_{T_n-}),$$

Thus, the cumulative jumps are equal to $\prod_{T_n \leq T} (1 + w\delta(T_n, S_{T_n-}))$.

When there are several risky assets, at time 0, the investor chooses to invest the initial weights w_i on the risky assets S_i and $(1 - \sum_{i=1}^d w_i)$ on the risk free asset with return e^{rT}. Thus we get:

$$R_T^{BH} = (1 - \sum_{i=1}^d w_i)e^{rT} + \sum_{i=1}^d w_i \frac{S_{i,T}}{S_{i,0}}.$$

For the constant mix strategy, at any time during the management period $[0, T]$, the investor chooses to invest the fixed weights w_i on the risky assets S_i and $(1 - \sum_{i=1}^d w_i)$ on the risk free asset with return e^{rT}. Therefore, we get:

$$\frac{dV_t^{CM}}{V_{t-}} =$$

$$(1-\sum_{i=1}^d w_i)rdt+\sum_{i=1}^d w_i \frac{dS_{i,t}}{S_{i,t}} = \left[(1 - \sum_{i=1}^d w_i)r + \sum_{i=1}^d w_i\mu_i(t, S_{i,t})\right] dt+\sum_{i=1}^d w_i\sigma_{i,t}dW_{i,t}+\sum_{i=1}^d w_i\delta_i dN_{i,t}.$$

Thus, using Ito's formula, we deduce that the constant mix portfolio value satisfies:

$$R_T^{CM} = \frac{V_T^{CM}}{V_0} = \tag{4}$$

$$\exp\left[\int_0^T \left[(1 - \sum_{i=1}^d w_i)r + \sum_{i=1}^d w_i\mu_i(t, S_{i,t}) - \frac{1}{2}\sum_{i=1}^d {w_i}^2\sigma_{i,t}^2 - \sum_{i<j}^d w_i w_j \sigma_{i,t}\sigma_{j,t}\rho_{i,j}\right] dt + \sum_{i=1}^d \int_0^T w_i\sigma_{i,t}dW_{i,t}\right]$$
$$\times \prod_{T_{i,n} \leq T, 1 \leq i \leq d} \left(1 + \sum_{i=1}^d w_i\delta_i(T_{i,n}, S_{T_{i,n}-})\right).$$

For the multidimensional geometric Brownian motion case, since the Brownian motions are functions of the risky asset prices, we can deduce[7]:

$$R_T^{CM} = \exp\left[\left[(1 - \sum_{i=1}^d w_i)r - \frac{1}{2}\sum_{i=1}^d (w_i^2 - w_i)\sigma_i^2 - \sum_{i<j}^d w_i w_j \sigma_i \sigma_j \rho_{i,j}\right] T\right] \prod_{i=1}^d \left(\frac{S_{i,T}}{S_{i,0}}\right)^{w_i}. \tag{5}$$

[7] For the geometric Brownian motion case, this formula appears in Perold and Sharpe (1988).

3 Comparison of buy-and-hold and constant mix portfolios

In what follows, we compare the two portfolio strategies by using main performance criteria.

3.1 Payoff comparison

We begin by comparing the terminal payoffs of both strategies in the GBM framework and for the one risky asset case. Such framework allows to focus only on the rebalancing effect. Therefore, the stock index price dynamics is given by the following stochastic process:

$$dS_t = S_t[\mu dt + \sigma dW_t], \tag{6}$$

where μ and σ are constant ($\sigma > 0$) and $\delta(t,s) = 0$. In that case, we deduce the two portfolio returns R_T^{BH} and R_T^{CM}:

$$R_T^{BH} = \frac{V_T^{BH}}{V_0} = (1-w)e^{rT} + w\exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma W_T\right], \tag{7}$$

$$R_T^{CM} = \frac{V_T^{CM}}{V_0} = \exp\left[\left[(1-w)r + w\mu - \frac{1}{2}w^2\sigma^2\right]T + w\sigma W_T\right]. \tag{8}$$

Denote by $R_T = S_T/S_0$ the risky asset return. We deduce:

$$R_T^{BH} = (1-w)e^{rT} + wR_T. \tag{9}$$

$$R_T^{CM} = \exp\left[\left[(1-w)r + \frac{\sigma^2}{2}\left(w - w^2\right)\right]T\right](R_T)^w \tag{10}$$

Therefore we have to compare $R_T^{BH} = (1-w)e^{rT} + wR_T$ with $\exp\left[\left[(1-w)r + \frac{\sigma^2}{2}\left(w-w^2\right)\right]T\right](R_T)^w$. Note that R_T^{BH} is always higher than $(1-w)e^{rT}$ while R_T^{CM} can reach 0 (for the theoretical point of view). The rebalancing return $R_T^R = R_T^{CM} - R_T^{BH}$ is given by:

$$R_T^R = \exp\left[\left[(1-w)r + \frac{\sigma^2}{2}\left(w-w^2\right)\right]T\right](R_T)^w - wR_T - (1-w)e^{rT}.$$

In what follows, recall that we consider the standard case $0 < w < 1$ corresponding to only long strategy on the risky asset.

The two payoffs intersect exactly at two values $R^{(1)}$ and $R^{(2)}$ of the risky return R_T.[8] We note also that the rebalancing return is R^R is maximal at

$$R^* = \exp\left[rT + w\frac{\sigma^2}{2}T\right]. \tag{11}$$

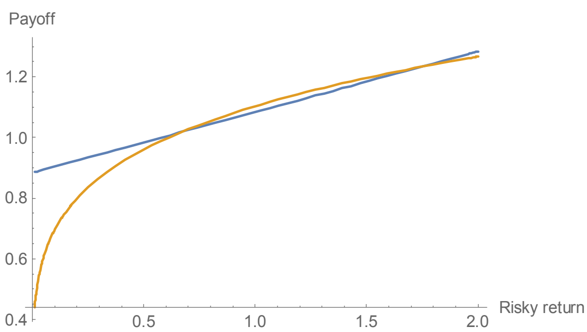
[8] For the special case $w = 0.5$, the two values $R^{(1)}$ and $R^{(2)}$ can be explicitly determined since they are solutions of a polynomial equation of order 2.

It does not depend on the trend μ of the the risky return R_T.[9]

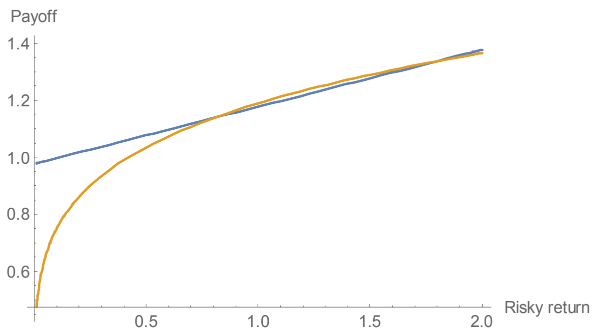
In what follows, to illustrate previous results, we choose the two following numerical base cases:

$$\begin{aligned} \text{Case 1} &: \quad \mu = 0.06; r = 0.01; \sigma = 0.15; T = 5, \\ \text{Case 2} &: \quad \mu = 0.12; r = 0.04; \sigma = 0.18; T = 5. \end{aligned}$$

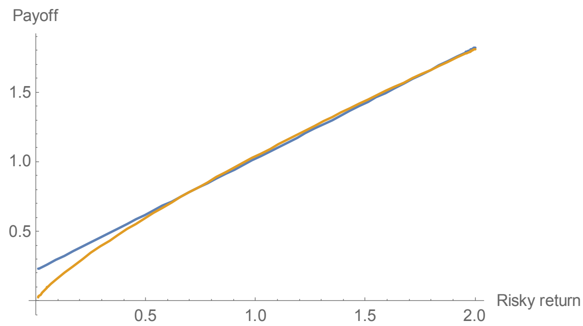
Figure (1) illustrates the comparison of the payoffs for both numerical cases 1 and 2. They always intersect at two points. Between these two values, the constant mix strategy provides a higher payoff.



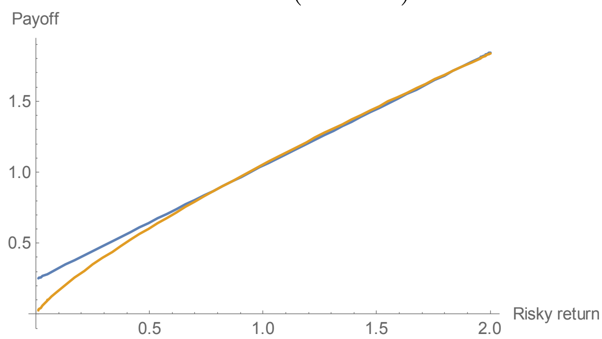
Case 1 ($w = 0.2$)



Case 2 ($w = 0.2$)



Case 1 ($w = 0.8$)



Case 2 ($w = 0.8$)

Figure 1: Payoffs of the two strategies as functions of the risky return for varying weight

[9] For the case $w < 0$ corresponding to be short on the risky asset (inverse leveraged fund), the diversification return R_T^D is a convex function w.r.t. the risky return R_T. The payoff of the constant mix strategy is higher than that of the buy-and-hold strategy if and only the risky return R_T lies outside the interval $\left[R^{(1)}, R^{(2)}\right]$.

For the case $w > 1$ corresponding to a leverage strategy, the diversification return R_T^D is a convex function w.r.t. the risky return R_T. The payoff of the constant mix strategy is higher than that of the buy-and-hold strategy if and only the risky return R_T lies outside the interval $\left[R^{(1)}, R^{(2)}\right]$.

For both previous cases, the diversification return is minimal at R^*.

We examine now the behavior of the rebalancing return. Since $0 < w < 1$, the rebalancing return R_T^R is a concave function w.r.t. the risky return R_T. The payoff of the constant mix strategy is higher than that of the buy-and-hold strategy if and only the risky return R_T lies inside the interval $\left[R^{(1)}, R^{(2)}\right]$. Figure 2 illustrates the rebalancing return as function of the risky return for both numerical cases 1 and 2. We investigate four values for weight w invested on the risky asset, namely $w = 0.2$; $0.4; 0.6$ and finally 0.8. We note that the numerical values of $R^{(1)}$ and $R^{(2)}$ do not depend significantly on the weight w. For the first case, the probability of getting a return of the constant mix strategy higher than that of the buy-and-hold is about 61% while for the second case, it is about 55%.

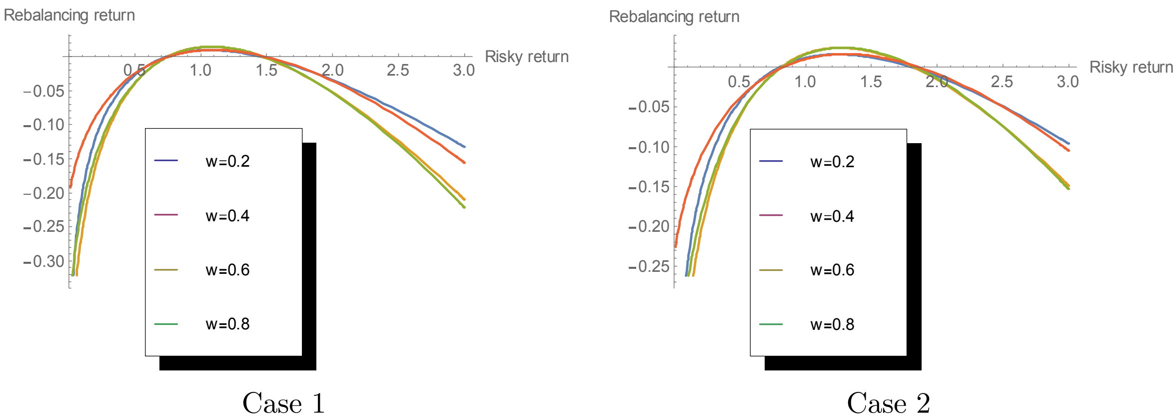
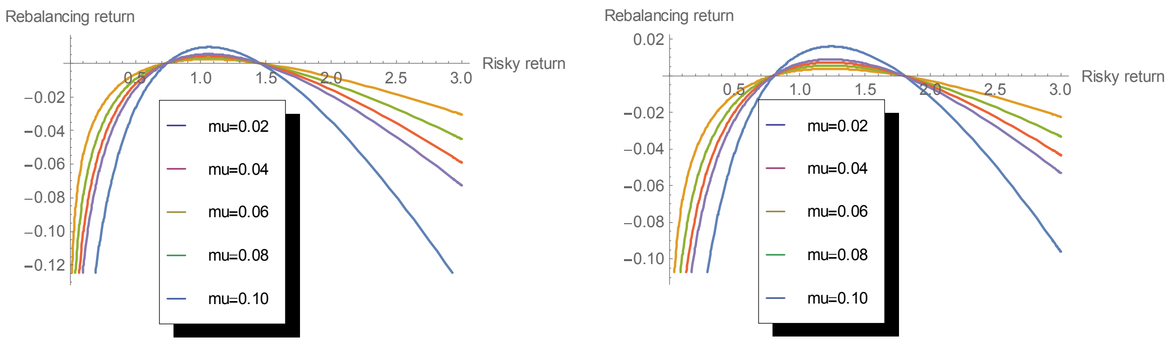


Figure 2: Rebalancing return as function of the risky return for varying weight

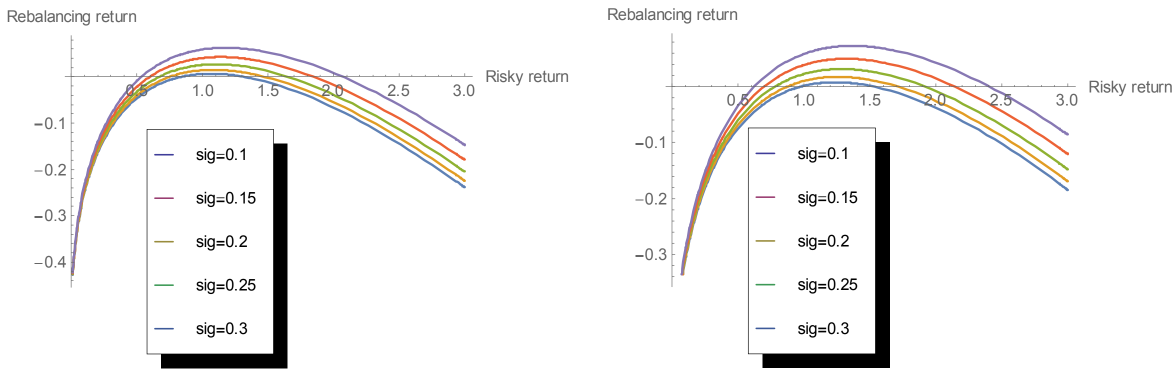
Figure 3 illustrates the rebalancing return as function of the risky return for varying trend values, namely $\mu = 0.02; 0.04; 0.06; 0.0.08; 0.1$. We fix $w = 0.5$. We note that the interval $\left[R^{(1)}, R^{(2)}\right]$ does not depend significantly on μ. Note also that, from Relation 11, we deduce that R^* does not depend on μ. However, the maximal value of the rebalancing return is increasing with respect to the trend.

Figure 4 illustrates the rebalancing return as function of the risky return for varying volatility values, namely $\sigma = 0.1; 0.15; 0.2; 0.25; 0.3$. We fix $w = 0.5$. We note that the interval $\left[R^{(1)}, R^{(2)}\right]$ is "increasing" w.r.t. the volatility σ; more precisely, the lower bound $R^{(1)}$ is decreasing while the upper bound $R^{(2)}$ is increasing w.r.t. σ. Note that, from Relation 11, we deduce that R^* is itself increasing w.r.t. σ. Note also that, for example, for case 1, the probability of getting a return of the constant mix strategy higher than that of the buy-and-hold is equal to about 61% for $\sigma = 0.15$ while it is equal to about 68% for $\sigma = 0.25$. For case 2, it is equal to about 47% for $\sigma = 0.15$ while it is equal to about 64% for $\sigma = 0.25$.



Case 1 Case 2

Figure 3: Rebalancing return as function of the risky return for varying trend



Case 1 Case 2

Figure 4: Rebalancing return as function of the risky return for varying volatility

Figure 5 illustrates the rebalancing return as function of the risky return for varying maturity T, namely $T = 1; 5; 10; 15; 30$ years. We fix $w = 0.5$. For case 1, we note that the lower bound $R^{(1)}$ is decreasing while the upper bound $R^{(2)}$ is increasing w.r.t. T . For numerical case 2, both bounds are increasing w.r.t. T. Note also that, from Relation 11, we deduce that R^* is itself increasing w.r.t. T. Looking at the probability of getting a return of the constant mix strategy higher than that of the buy-and-hold is also decreasing w.r.t. T, we get for example: for case 1, it is equal to about 61% for $T = 5$ while it is equal to about 54% for $T = 10$; for case 2, it is equal to about 55% for $T = 5$ while it is equal to about 44% for $T = 10$.

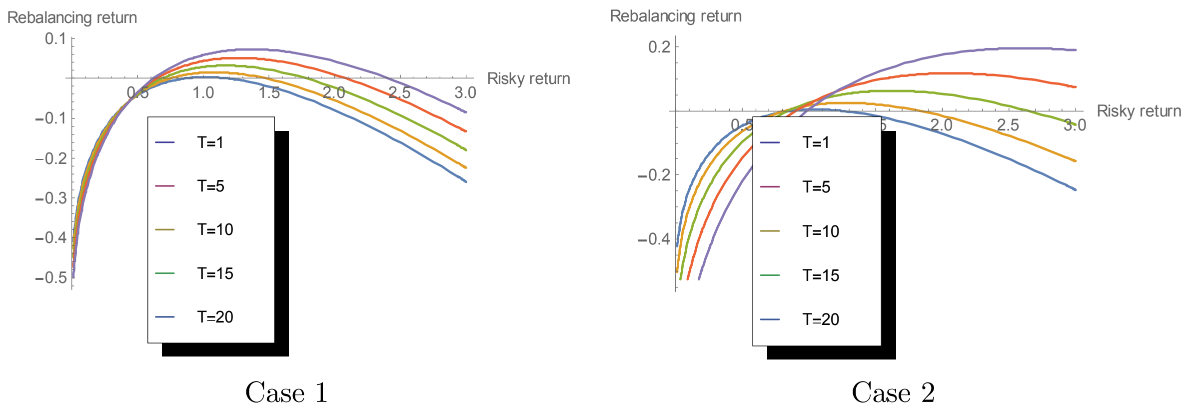


Figure 5: Rebalancing return as function of the risky return for varying maturity

Examine now the probability that constant mix return is higher than buy-and-hold return. Tables (1) and (2) illustrate the results for the GBM case for various financial parameters. Here we consider also two cases a and b corresponding to interest rate values respectively equal respectively to 1% and 3%. Looking at Tables (1) and (2), we note that the probability that constant mix return is higher than buy-and-hold return is most of the time between 60% and 70%, more precisely around 66% as emphasized by Wise (1996). This probability is smaller than 0.5 only for low volatility and high trend. It is increasing with respect to volatility σ and decreasing with respect to trend μ. It is also increasing with respect to the weight w invested on the risky asset. Finally, we remark that it is slightly increasing with respect to the interest rate and decreasing with respect to time horizon T.[10]

[10] Other detailed illustrations of such properties are available upon the authors on request.

Table 1: Probability that constant mix return is higher than buy-and-hold return, T=5

$w = 0.2$	$\mu = 5,00\%$		$\mu = 6,00\%$		$\mu = 7,00\%$		$\mu = 8,00\%$		$\mu = 9,00\%$	
σ	Case a	Case b	Case a	Case b	Case a	Case b	Case a	Case b	Case a	Case b
15%	63.40%	67.75%	59.89%	66.06%	55.67%	63.40%	50.91%	59.89%	45.80%	55.67%
20%	66.77%	68.44%	65.08%	67.90%	62.86%	66.77%	60.18%	65.08%	57.10%	62.86%
25%	68.18%	68.57%	67.42%	68.57%	66.28%	68.18%	64.79%	67.41%	62.98%	66.28%
30%	68.75%	68.43%	68.51%	68.73%	68%	68.75%	67.23%	68.51%	66.21%	68.00%
$w = 0.5$	$\mu = 5,00\%$		$\mu = 6,00\%$		$\mu = 7,00\%$		$\mu = 8,00\%$		$\mu = 9,00\%$	
σ	Case a	Case b	Case a	Case b	Case a	Case b	Case a	Case b	Case a	Case b
15%	64.07%	67.97%	60.74%	66.51%	56.67%	64.07%	52.02%	60.74%	46.96%	56.67%
20%	67.27%	68.47%	65.79%	68.16%	63.78%	67.27%	61.28%	65.80%	58.35%	63.78%
25%	68.43%	68.34%	67.90%	68.58%	67.00%	68.43%	65.73%	67.90%	64.12%	67.00%
30%	68.68%	67.88%	68.68%	68.42%	68.42%	68.68%	67.88%	68.68%	67.09%	68.42%
$w = 0.8$	$\mu = 5,00\%$		$\mu = 6,00\%$		$\mu = 7,00\%$		$\mu = 8,00\%$		$\mu = 9,00\%$	
σ	Case a	Case b	Case a	Case b	Case a	Case b	Case a	Case b	Case a	Case b
15%	64.72%	68.17%	61.57%	66.94%	57.65%	64.71%	53.12%	61.57%	48.14%	57.65%
20%	67.72%	68.44%	66.48%	68.38%	64.68%	67.72%	62.37%	66.48%	59.60%	64.68%
25%	68.60%	68.02%	68.32%	68.51%	67.65%	68.61%	66.61%	68.32%	65.21%	67.65%
30%	68.50%	67.23%	68.75%	68.00%	68.72%	68.51%	68.43%	68.75%	67.87%	68.73%

Table 2: Probability that constant mix return is higher than buy-and-hold return, T=10

$w = 0.2$	$\mu = 5,00\%$		$\mu = 6,00\%$		$\mu = 7,00\%$		$\mu = 8,00\%$		$\mu = 9,00\%$	
σ	Case a	Case b	Case a	Case b	Case a	Case b	Case a	Case b	Case a	Case b
15%	58.88%	67.23%	52.52%	63.92%	45.33%	58.88%	37.84%	52.52%	30.52%	45.33%
20%	65.31%	68.62%	62.04%	67.54%	57.88%	65.31%	53.04%	62.04%	47.71%	57.88%
25%	68.10%	68.88%	66.57%	68.87%	64.36%	68.09%	61.51%	66.57%	58.12%	64.36%
30%	69.23%	68.61%	68.74%	69.19%	67.72%	69.23%	66.20%	68.74%	64.21%	67.72%
$w = 0.5$	$\mu = 5,00\%$		$\mu = 6,00\%$		$\mu = 7,00\%$		$\mu = 8,00\%$		$\mu = 9,00\%$	
σ	Case a	Case b	Case a	Case b	Case a	Case b	Case a	Case b	Case a	Case b
15%	60.13%	67.68%	54.02%	64.81%	46.97%	60.13%	39.51%	54.02%	32.12%	46.97%
20%	66.29%	68.67%	63.42%	68.07%	59.61%	66.29%	55.02%	63.42%	49.86%	59.61%
25%	68.60%	68.41%	67.55%	68.89%	65.76%	68.60%	63.31%	67.55%	60.25%	65.76%
30%	69.10%	67.51%	69.10%	68.57%	68.57%	69.10%	67.51%	69.10%	65.95%	68.57%
$w = 0.8$	$\mu = 5,00\%$		$\mu = 6,00\%$		$\mu = 7,00\%$		$\mu = 8,00\%$		$\mu = 9,00\%$	
σ	Case a	Case b	Case a	Case b	Case a	Case b	Case a	Case b	Case a	Case b
15%	61.35%	68.07%	55.53%	65.65%	48.65%	61.35%	41.23%	55.53%	33.78%	48.65%
20%	67.19%	68.62%	64.75%	68.50%	61.30%	67.19%	57.00%	64.75%	52.04%	61.30%
25%	68.95%	67.78%	68.38%	68.75%	67.05%	68.95%	65.02%	68.38%	62.33%	67.05%
30%	68.74%	66.20%	69.23%	67.72%	69.19%	68.74%	68.61%	69.23%	67.51%	69.19%

In what follows, we focus on the impact of potential jumps. For the one risky asset case and for a pure jump process, we deduce the value of the rebalancing return:

$$R_T^R =$$

$$\exp\left[(1-w)rT\right] \prod_{T_n \leq T} (1 + w\delta(T_n, S_{T_n-})) - (1-w)e^{rT} - w \prod_{T_n \leq T} (1 + \delta(T_n, S_{T_n-}))$$

We note that, for $r \simeq 0$, for the buy-and-hold strategy, the cumulative jumps yield to $(1-w)+w \prod_{T_n \leq T} (1 + \delta(T_n, S_{T_n-}))$ while, for the constant mix portfolio, we get $\prod_{T_n \leq T} (1 + w\delta(T_n, S_{T_n-}))$. Obviously, R^R is equal to 0 if there is only one period or if the weight w is equal to 1. Then, if all relative jumps $\delta(T_n, S_{T_n-})$ have the same sign, we have $R_T^R \leq 0$ or equivalently $R_T^{CM} \leq R_T^{BH}$. Otherwise, we may have $R_T^{CM} \geq R_T^{BH}$. Previous results are in accordance with what happens when there is no jump. Indeed, looking at the Greek Gamma of the portfolio return in the GBM case, we get:

$$\Gamma = \frac{\partial R_T^{CM}}{\partial R_T} = w\,(w-1) \exp\left[\left[(1-w)r + \frac{\sigma^2}{2}\left(w - w^2\right)\right] T\right] (R_T)^{w-2}.$$

Therefore, since $0 \leq w \leq 1$, we have $\Gamma \leq 0$. Thus it means that, when the risky asset rises, you have to sell, while, when it falls, you have to buy. It is exactly what happens with previous jump case. This prevents in particular to benefit significantly from market rises.

When dealing with several risky assets, using Relation (4), we deduce that the rebalancing return for the GBM case is given by:

$$R_T^R = \lambda \prod_{i=1}^{d} \left(\frac{S_{i,T}}{S_{i,0}}\right)^{w_i} - (1 - \sum_{i=1}^{d} w_i)e^{rT} - \sum_{i=1}^{d} w_i \frac{S_{i,T}}{S_{i,0}}, \tag{12}$$

with

$$\lambda = \exp\left[\left[(1 - \sum_{i=1}^{d} w_i)r - \frac{1}{2}\sum_{i=1}^{d}(w_i^2 - w_i)\sigma_i^2 - \sum_{i<j}^{d} w_i w_j \sigma_i \sigma_j \rho_{i,j}\right] T\right].$$

For the pure jumps case, we deduce:

$$\begin{aligned} R_T^R &= \exp\left[(1-w)rT\right] \prod_{T_{i,n} \leq T, 1 \leq i \leq d} \left(1 + \sum_{i=1}^{d} w_i \delta_i(T_{i,n}, S_{T_{i,n}-})\right) \\ &\quad -(1-w)e^{rT} - w \prod_{T_{i,n} \leq T, 1 \leq i \leq d} \left(1 + \sum_{i=1}^{d} \delta_i(T_{i,n}, S_{T_{i,n}-})\right) \end{aligned}$$

Therefore, for this latter case, the cumulative jumps yield to

$$(1-w)+w\prod_{T_{i,n}\leq T,1\leq i\leq d}\left(1+\sum_{i=1}^{d}\delta_i(T_{i,n},S_{T_{i,n}-})\right)$$

while, for the constant mix portfolio, we get $\prod_{T_{i,n}\leq T,1\leq i\leq d}\left(1+\sum_{i=1}^{d}w_i\delta_i(T_{i,n},S_{T_{i,n}-})\right)$.

3.2 First four moments

We study now the first four moments of both strategies. We begin by establishing general results about the comparison of their four moments, extending first previous results about expectation and variance in the i.i.d. case (see e.g. Wise, 1996) and second providing the comparison of their skewness and kurtosis.[11] For this purpose, we use the risky dynamics considered in Relation (??), which corresponds to a jump-diffusion process where both drift and volatility are deterministic but not necessarily constant, namely:

$$dS_t = S_t[\mu(t,S_t)dt + \sigma_t dW_t + \delta(t,S_t)dN_t],$$

where $\mu(t,s)$ is a function defined on $[0,T]\times\mathbb{R}^+$ with values in $\mathbb{R}$ and σ_t ($\sigma>0$) is a stochastic process, W denotes a standard Brownian motion with respect to a given filtration $(\mathcal{F}_t)_t$ and N is a point process.

Comparison of expectations: assume that both the drift $\mu(.)$ and the volatility $\sigma(.)$ are deterministic. Assume also that N is a compound Poisson process with intensity λ. Denote by $\overline{\delta}$ the common expectation of the relative jumps of the risky asset $\frac{\Delta S_{T_n}}{S_{T_n-}}=\delta(T_n,S_{T_n-})$.at jump times T_n. Then we get:

$$E\left[R_T^{BH}\right]=(1-w)e^{rT}+w\exp\left[\int_0^T\mu(t)dt+\overline{\delta}\lambda T\right], \tag{13}$$

$$E\left[R_T^{CM}\right]=\exp\left[(1-w)rT+w\left(\int_0^T\mu(t)dt+\overline{\delta}\lambda T\right)\right]. \tag{14}$$

Therefore, by convexity of the exponential function, we deduce that $E\left[R_T^{CM}\right]<E\left[R_T^{BH}\right]$.

Previous results extends the result of Cheng and Deets (1971) and Wise (1996) who compare the performances of buy-and-hold and rebalancing strategies assuming that risky asset prices follow random walks (i.e. are i.i.d.).

Comparison of variances:

Denote by $\overline{\delta^2}$ the common expectation of the squares of the relative jumps of the risky asset $\frac{\Delta S_{T_n}}{S_{T_n-}}=\delta(T_n,S_{T_n-})$.at jump times T_n. Then we get:

$$Variance\left[R_T^{BH}\right]=w^2\exp\left[2\left(\int_0^T\mu(t)dt+\overline{\delta}\lambda T\right)\right]\left(e^{\int_0^T\sigma^2(t)dt+\overline{\delta^2}\lambda T}-1\right), \tag{15}$$

[11] See Appendix B for proofs of all the results of this section.

$$Variance\left[R_T^{CM}\right] = \exp\left[2\left((1-w)rT + w\left(\int_0^T \mu(t)dt + \overline{\delta}\lambda T\right)\right)\right]\left(e^{w^2\left(\int_0^T \sigma^2(t)dt + \overline{\delta^2}\lambda T\right)} - 1\right). \tag{16}$$

Therefore, we deduce that $Variance\left[R_T^{CM}\right] \leq Variance\left[R_T^{BH}\right]$.

Comparison of skewness and kurtosis:

Denote $s_T^2 = \int_0^T \sigma^2(t)dt$. When there is no jump, the skewness and excess kurtosis are respectively equal to:

$$Sk\left[R_T^{BH}\right] = \sqrt{e^{s_T^2} - 1}\left(2 + e^{s_T^2}\right) \text{ and } Sk\left[R_T^{CM}\right] = \sqrt{e^{w^2 s_T^2} - 1}\left(2 + e^{w^2 s_T^2}\right), , \tag{17}$$

and

$$ExcessKurt\left[R_T^{BH}\right] = \left(e^{4s_T^2} + 2e^{3s_T^2} + 3e^{2s_T^2} - 6\right), \tag{18}$$

$$ExcessKurt\left[R_T^{CM}\right] = \left(e^{4w^2 s_T^2} + 2e^{3w^2 s_T^2} + 3e^{2w^2 s_T^2} - 6\right). \tag{19}$$

Looking at previous relations, we can see that, to compare both skewness and excess kurtosis, we have just to compare $Sk\left[R_T^{CM}\right]$ and $ExcessKurt\left[R_T^{CM}\right]$ for $w = 1$ since both functions are increasing w.r.t. w while they are constant w.r.t. w for the BH strategy. We deduce that $Sk\left[R_T^{CM}\right] \leq Sk\left[R_T^{BH}\right]$ and $ExcessKurt\left[R_T^{CM}\right] \leq ExcessKurt\left[R_T^{BH}\right]$. Note also that they are increasing w.r.t. the volatility $\sigma(t)$ and the time horizon T. Table (3) illustrates the comparison of both expectations and variances. We set $r = 1\%$ and $T = 5$ years. We consider three cases for the weight invested on the risky asset, namely $w = 0.2$, $w = 0.5$ and $w = 0.8$. For each case, the first lines of Table (3) correspond to expectations values.

As can be deduced form Relations (13), (14), (15), and (16), expectations are increasing with respect to the interest rate r, to the trend μ, to the time horizon T and to the weight w invested on the risky asset provided that the trend is higher than the interest rate. Variances satisfy the same properties. Additionally, they are both increasing with respect to the volatility σ. Note that the expectations of both returns do not depend on the volatility σ.

Table 3: Expectation and standard deviation of constant mix and buy-and-hold strategies

$w = 0.2$	$\mu = 5,00\%$		$\mu = 6,00\%$		$\mu = 7,00\%$		$\mu = 8,00\%$		$\mu = 9,00\%$	
σ	CM	BH	CM	BH	CM	BH	CM	BH	CM	BH
	9.42%	9.78%	10.52%	11.10%	11.63%	12.78%	12.75%	13.93%	13.88%	15.47%
15%	7.35%	8.86%	7.42%	9.31%	7.49%	9.79%	7.57%	10.29%	7.64%	10.82%
20%	9.80%	12.08%	9.90%	12.70%	10.04%	13.35%	10.10%	14.04%	10.20%	14.76%
25%	12.27%	15.55%	12.39%	16.35%	12.52%	17.19%	12.64%	18.07%	12.77%	19.00%
30%	14.74%	19.36%	14.89%	20.35%	15.04%	21.40%	15.19%	22.50%	15.35%	23.65%
$w = 0.5$	$\mu = 5,00\%$		$\mu = 6,00\%$		$\mu = 7,00\%$		$\mu = 8,00\%$		$\mu = 9,00\%$	
σ	CM	BH	CM	BH	CM	BH	CM	BH	CM	BH
	16.18%	16.76%	19.12%	20.05%	22.14%	23.51%	25.23%	27.15%	28.40%	30.97%
15%	19.62%	22.15%	20.12%	23.29%	20.63%	24.48%	21.15%	25.74%	21.68%	27.06%
20%	26.30%	30.21%	26.97%	31.76%	27.65%	33.38%	28.35%	35.09%	29.07%	36.89%
25%	33.12%	38.88%	33.96%	40.87%	34.82%	42.97%	35.70%	45.18%	36.60%	47.49%
30%	40.09%	48.40%	41.10%	50.88%	42.15%	53.48%	43.21%	56.23%	44.31%	59.12%
$w = 0.8$	$\mu = 5,00\%$		$\mu = 6,00\%$		$\mu = 7,00\%$		$\mu = 8,00\%$		$\mu = 9,00\%$	
σ	CM	BH	CM	BH	CM	BH	CM	BH	CM	BH
	23.37%	23.74%	28.40%	29.01%	33.64%	34.55%	39.09%	40.37%	44.77%	46.49%
15%	33.71%	35.44%	35.08%	37.26%	36.51%	39.17%	38.00%	41.18%	39.55%	43.29%
20%	45.58%	48.33%	47.45%	50.81%	49.38%	53.42%	51.40%	56.15%	53.50%	59.03%
25%	58.05%	62.21%	60.42%	65.41%	62.88%	68.76%	65.45%	72.28%	68.12%	76.00%
30%	71.27%	77.44%	74.18%	81.41%	77.21%	85.58%	80.36%	89.97%	86.64%	94.583%

The multidimensional case. In what follows, we compare the expectations and variances of both strategies. We examine also the impact of correlations. We assume that the drifts $\mu_i(.)$ and the volatility $\sigma_i(.)$ are deterministic and that there is no jump.

Recall that we have:

$$R_T^{BH} = (1 - \sum_{i=1}^{d} w_i)e^{rT} + \sum_{i=1}^{d} w_i \frac{S_{i,T}}{S_{i,0}}.$$

$$R_T^{CM} =$$

$$\exp\left[\int_0^T \left[(1 - \sum_{i=1}^{d} w_i)r + \sum_{i=1}^{d} w_i\mu_{i,t} - \frac{1}{2}\sum_{i=1}^{d} {w_i}^2\sigma_{i,t}^2 - \sum_{i<j}^{d} w_i w_j \sigma_{i,t}\sigma_{j,t}\rho_{i,j}\right] dt + \sum_{i=1}^{d}\int_0^T w_i\sigma_{i,t}dW_{i,t}\right]$$

Therefore, we get:

$$E\left[R_T^{BH}\right] = (1 - \sum_{i=1}^{d} w_i)e^{rT} + \sum_{i=1}^{d} w_i \exp\left[\int_0^T \mu_{i,t}dt\right], \tag{20}$$

$$E\left[R_T^{CM}\right] = \exp\left[(1 - \sum_{i=1}^{d} w_i)rT + \sum_{i=1}^{d} w_i \left(\int_0^T \mu_{i,t}dt\right)\right]. \tag{21}$$

Therefore, by convexity of the exponential function, we deduce that $E\left[R_T^{CM}\right] < E\left[R_T^{BH}\right]$.

To compare the variances, consider the particular case with no riskless asset and $\mu_{i,t} = \mu_{j,t}$ for all i, j. In that case, to compare the variances, we have just to compare

$$\sum_{i=1}^{d} w_i^2 \left(e^{\int_0^T \sigma_{i,t}^2 dt} - 1 \right) + 2 \sum_{1 \leq i < j \leq d} w_i w_j \left(\exp\left[\int_0^T \sigma_{i,t}\sigma_{j,t}\rho_{i,j} dt \right] - 1 \right)$$

$$\text{with } \left(e^{\sum_{i=1}^{d} w_i^2 \left(\int_0^T \sigma_{i,t}^2 dt\right) + 2\sum_{1 \leq i < j \leq d} w_i w_j \int_0^T \sigma_{i,t}\sigma_{j,t}\rho_{i,j} dt} - 1 \right).$$

Using the convexity of e^{x-1}, we have for any $(a_l)_{1 \leq l \leq m}$ satisfying $a_i \geq 0$ and $\sum_{l=1}^{m} a_l = 1$,

$$\exp\left[\sum_{i=1}^{m} a_l x_l \right] - 1 \leq \sum_{i=1}^{m} a_l \left(\exp\left[x_l\right] - 1 \right)$$

Note that

$$\sum_{i=1}^{d} w_i^2 + 2 \sum_{1 \leq i < j \leq d} w_i w_j = \left(\sum_{i=1}^{d} w_i \right)^2 = 1.$$

Therefore, we deduce that $Variance\left[R_T^{CM}\right] \leq Variance\left[R_T^{BH}\right]$. Note that, if we consider the equally weighting case and assume that variances and correlations are equal, then the difference between the two variances converges to 0 as the number d of assets converges to infinity.

Examine now the impact of correlations. Obviously, the correlation has no impact on the comparison of expected returns. For the variances, let us examine the difference $Variance\left[R_T^{BH}\right] - Variance\left[R_T^{CM}\right]$ for the case $d = 2$ and equal drifts. Denote:

$$\rho^* = \frac{w^2 \left(\int_0^T \sigma_{1,t}^2 dt \right) + (1-w)^2 \left(\int_0^T \sigma_{2,t}^2 dt \right)}{\left(\int_0^T \sigma_{1,t}\sigma_{2,t} dt \right) \left[1 - 2w(1-w)\right]} \tag{22}$$

The difference $Variance\left[R_T^{BH}\right] - Variance\left[R_T^{CM}\right]$ reaches a minimum at ρ^* if condition $\rho^* \leq 1$ is satisfied. Note that condition $\rho^* \leq 1$ is equivalent to:

$$\left(w\sqrt{\int_0^T \sigma_{1,t}^2 dt} + (1-w)\sqrt{\int_0^T \sigma_{2,t}^2 dt} \right)^2 \leq \int_0^T \sigma_{1,t}\sigma_{2,t} dt.$$

The previous condition is not always satisfied. It depends on the choice of the parameters w, $\sigma_{1,t}$ and $\sigma_{2,t}$. Therefore, we must distinguish two cases:

Case 1: $\rho^* \geq 1$. In that case, the difference $Variance\left[R_T^{BH}\right] - Variance\left[R_T^{CM}\right]$ is decreasing with respect to the correlation ρ.

Case 2: $\rho^* < 1$. In that case, the difference $Variance\left[R_T^{BH}\right] - Variance\left[R_T^{CM}\right]$ is first decreasing then increasing with respect to the correlation ρ. It means that, at $\rho = \rho^*$, the advantage of the constant mix strategy is all the weaker from the point of view of variance.

3.3 Cumulative distribution function (CDF) of the two portfolio strategies

For the Brownian geometric case, we can determine explicitly the cumulative distribution functions, which allows for example that there is no stochastic dominance of one strategy against the other one even at the second order.

Denote by Φ the cdf of the standard Gaussian distribution, namely $\Phi(x) = \frac{1}{2\sqrt{\pi}} \int_0^x \exp(-u^2/2)du$. Using standard calculus, we get:

$$F_{R_T^{BH}}(x) = \Phi\left[\left(Log\left(\frac{x-(1-w)e^{rT}}{w}\right) - (\mu - \frac{1}{2}\sigma^2)T\right)/\sigma\sqrt{T}\right]$$

$$F_{R_T^{CM}}(x) = \Phi\left[\left(Log\,(x) - \left[(1-w)r + w\mu - \frac{1}{2}w^2\sigma^2\right]T\right)/\left(w\sigma\sqrt{T}\right)\right]$$

Thus to compare the cdfs of both portfolio strategies, it is necessary and sufficient to compare $\left(Log\left(\frac{x-(1-w)e^{rT}}{w}\right) - (\mu - \frac{1}{2}\sigma^2)T\right)/\sigma\sqrt{T}$ with $\left(Log\,(x) - \left[(1-w)r + w\mu - \frac{1}{2}w^2\sigma^2\right]T\right)/\left(w\sigma\sqrt{T}\right)$.

This is equivalent to the comparison of $w\left(Log\left(\frac{x-(1-w)e^{rT}}{w}\right) - (\mu - \frac{1}{2}\sigma^2)T\right)$ with $\left(Log\,(x) - \left[(1-w)r + w\mu - \frac{1}{2}w^2\sigma^2\right]T\right)$.

Consider the following equation: find x such that

$$w\left(Log\left(\frac{x-(1-w)e^{rT}}{w}\right) - (\mu - \frac{1}{2}\sigma^2)T\right) = \left(Log\,(x) - \left[(1-w)r + w\mu - \frac{1}{2}w^2\sigma^2\right]T\right).$$

This is equivalent to:

$$w\left(Log\left(\frac{x-(1-w)e^{rT}}{w}\right) + \frac{1}{2}\sigma^2T\right) = \left(Log\,(x) - \left[(1-w)r - \frac{1}{2}w^2\sigma^2\right]T\right),$$

and also to:

$$wLog\left(\frac{x-(1-w)e^{rT}}{w}\right) + \frac{1}{2}\sigma^2w(1-w)T = Log\,(x) - (1-w)rT. \quad (23)$$

We note that the drift μ plays no role in previous Equation 23.

We can prove that the two cdf curves intersect exactly at two points (See Appendix). This property implies that none of the two portfolio strategies dominates the other one at the first order stochastic dominance. There is also no stochastic dominance at the second order since:

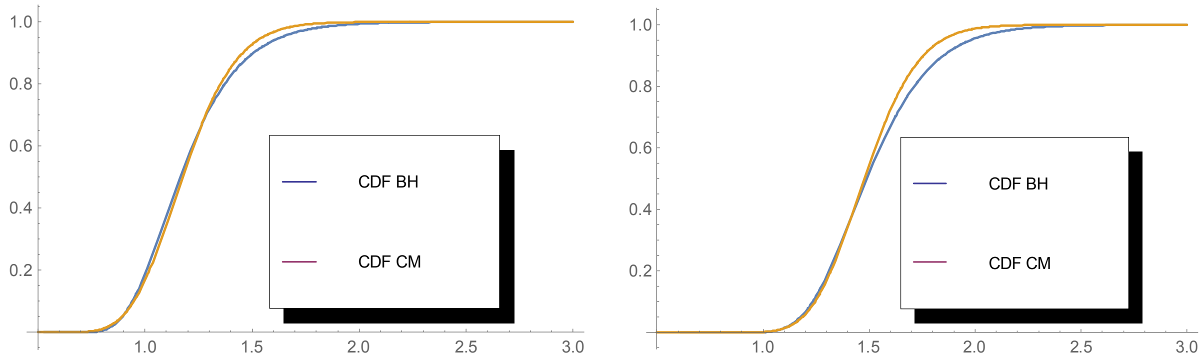
1) We have: $F_{R_T^{CM}}(x) > F_{R_T^{BH}}(x) = 0$ for $x < 1 - e^{rT}$, implying that R_T^{CM} cannot dominate R_T^{BH} at the second order

2) For any weight w satisfying $0 < w < 1$, there exists a CRRA utility for which this weight is optimal when maximizing the expected utility, , implying that R_T^{BH} cannot dominate R_T^{CM} at the second order.

We still choose the two following numerical base cases: (1) $\mu = 0.06; r = 0.01; \sigma = 0.15; T = 5$, (2) $\mu = 0.12; r = 0.04; \sigma = 0.18; T = 5$.

Figure (6) illustrates the comparison of the cumulative distributions functions with the two intersection points. We note that, for small and high x values, we have $F_{R_T^{CM}}(x) > F_{R_T^{BH}}(x)$.

However, the two curves are rather close. Note that, for longer time horizons, the difference between the two is more pronounced.



Case 1

Case 2

Figure 6: Comparison of cdf

3.4 Comparison by means of Sharpe ratio and Kappa performance measures

The Sharpe ratio is the very well known performance measure introduced by Sharpe (1996, 1994). For a given asset return X and a riskfree return R_f, it is defined as follows:

$$Sharpe\left(X\right) = \frac{\mathbb{E}_{\mathbb{P}}\left[X\right] - R_f}{\sqrt{Variance_{\mathbb{P}}\left[X\right]}}. \tag{24}$$

The Omega performance measure has been first introduced by Keating and Shadwick (2002) and Cascon *et al.* (2003). It is designed to overcome the shortcomings of performance measures based only on the mean and the variance of the distribution of the returns. Omega measure takes account of the entire return distribution while requiring no parametric assumption on the distribution. The returns both below and above a given loss threshold are considered. More precisely, Omega is defined as the probability weighted ratio of gains to losses relative to a return threshold. The exact mathematical definition is given by:

$$\Omega_X(L) = \frac{\int_L^b \left(1 - F\left(x\right)\right) dx}{\int_a^L F\left(x\right) dx}, \tag{25}$$

where $F(.)$ is the cumulative distribution function of the asset return X defined on the interval (a, b), with respect to the probability distribution $\mathbb{P}$ and L is the return threshold selected by the investor. For any investor, returns below her loss threshold are considered as losses and returns above as gains. At a given return threshold, the investor should always prefer the portfolio with the highest value of Omega.

The Omega function exhibits the following properties:

- First, as shown for example in Kazemi *et al.* (2004), Omega can be written as:

$$\Omega_X(L) = \frac{\mathbb{E}_{\mathbb{P}}\left[(X-L)^+\right]}{\mathbb{E}_{\mathbb{P}}\left[(L-X)^+\right]}. \tag{26}$$

It is the ratio of the expectations of gains above the threshold L to the expectations of the losses below the threshold L.[12]

Kazemi *et al.* (2004) define the Sharpe Omega measure as:

$$Sharpe_{\Omega}(L) = \frac{\mathbb{E}_{\mathbb{P}}[X] - L}{\mathbb{E}_{\mathbb{P}}\left[(L-X)^+\right]} = \Omega_X(L) - 1. \tag{27}$$

Note that if $\mathbb{E}_{\mathbb{P}}[X] < L$, the Sharpe Omega will be negative, otherwise it will be positive.

- For $L = \mathbb{E}_{\mathbb{P}}[X]$, $\Omega_X(L) = 1$,
- $\Omega_X(.)$ is a monotone decreasing function.
- $\Omega_X(.) = \Omega_Y(.)$ if and only if $F_X = F_Y$.

Typically, consider a strategy which consists in investing 100% of the initial amount in the risky asset. In that case, the portfolio payoff is equal to the stock payoff S at time T which is modelled by a geometric Brownian motion. Therefore, here we have $X = S_0 \exp[(\mu - \sigma^2/2)T + \sigma W_T]$, where W_T has the Gaussian distribution $\mathcal{N}(0,T)$. Then, $\mathbb{E}_{\mathbb{P}}[X] = S_0 \exp[\mu T]$ does not depend on the volatility. Thus, if $S_0 \exp[\mu T] < L$ then the Sharpe Omega is an increasing function of the volatility σ (due to the Vega of the put option). If $S_0 \exp[\mu T] > L$, the Sharpe Omega is a decreasing function of the volatility σ.

The level must be specified exogenously. It varies according to investment objective and individual risk aversion. As proved by Unser (2002), we are often only interested in an evaluation of outcomes which are "risky", i.e. their values are smaller than a given target, thus reflecting the attitude towards downside risk. Examples would be an inflation rate for pension incomes, or the rate of a benchmark financial index (see Bertrand and Prigent (2011) for an application to portfolio insurance). Omega performance measure and portfolio insurance. Such downside risk measures have been examined for instance in Ebert (2005), and are linked to the measures proposed by Fishburn (1977, 1984).

[12] Kazemi *et al.* (2004) note that, by multiplying both numerator and denominator by the discount factor, Omega can be considered as the ratio of the prices of a call option to a put option written on X with strike price L but both evaluated under the historical probability $\mathbb{P}$ instead of the risk neutral one. For example, if the risky asset S follows a GBM, then, mathematically speaking, the Omega value of a whole investment in S is the ratio of the Black-Scholes call value upon the put value with strike L and value of the drift of S instead of the riskless return.

In fact, the Sharpe Omega measure is one of the Kappa measures considered in Kaplan and Knowles (2004). These latter ones are defined by: for $l = 1, 2, ...,$

$$Kappa_l\,(L) = \frac{\mathbb{E}_{\mathbb{P}}\,[X] - L}{\left(\mathbb{E}_{\mathbb{P}}\left[\left[(L - X)^+\right]^l\right]\right)^{\frac{1}{l}}}. \tag{28}$$

For $l = 1$, we get the Sharpe Omega measure and, for $l = 2$, we recover the Sortino ratio. Zakamouline (2010) proves that Kappa measures correspond to performance measures based on piecewise linear plus power utility functions. To prove such result, consider the following utility function:

$$U_L(v) = (v - L)^+ - (\left[(L - v)^+\right] + \frac{\Phi}{n}\left[(L - v)^+\right]^n),$$

with $\Phi > 1$ and n a nonzero integer. Then, as shown in Zakamouline (2014), the investor's capital allocation problem yields to the following relation:

$$E[U^*(V)] = \frac{n-1}{n}\left(\frac{E[V] - L}{(E[(L - V)^{+n}])^{\frac{1}{n}}}\right)^{\frac{n}{n-1}},$$

where $E[U^*(V)]$ denotes the utility of the optimal allocation. Therefore, $E[U^*(V)]$ is an increasing transformation of the Kappa(n) ratio, which proves that this latter one is based on the utility U_L. Note that U_l is convex on $]-\infty, L]$ if and only if $n = 1$. This corresponds to the Omega measure, which is a limiting case when $n \to 1$. Therefore, the Omega measure is linked to the maximization of an expected utility with loss aversion, as introduced by Tversky and Kahneman (1992).

Tables (4) and (5) display respectively the Sharpe ratio and the Sharpe Omega ratio of both portfolio strategies (we choose $L = 0\%$). We set $r = 1\%$ and $T = 5$ years. We consider three cases for the weight invested on the risky asset, namely $w = 0.2$, $w = 0.5$ and $w = 0.8$. As emphasized by for example Boscaljon et al. (2008), rebalancing induces higher Sharpe ratios than buy-and-hold. However, the difference is small. Obviously, the Sharpe ratio is increasing with respect to the trend μ and decreasing with respect to the volatility σ. Note also that the Sharpe ratio of the buy-and-hold strategy does not depend on the weight w. The Sharpe ratio of the rebalancing strategy is slightly decreasing with respect to the weight w. Contrary to the Sharpe ratio case, rebalancing induces smaller Sharpe Omega ratios than buy-and-hold strategy. This is due to the impact of loss aversion which is taken into account by the Sharpe Omega ratio. However, for high values of the weight (0.8 for example), the two Sharpe Omega ratios are quite close. Indeed, recall that, for $w = 1$, the two strategies are equal. The Sharpe Omega ratio is increasing with respect to the trend μ and decreasing with respect to the volatility σ. It is slightly decreasing with respect to the weight w.

Table 4: Sharpe ratio of constant mix and buy-and-hold strategies

$w = 0.2$	$\mu = 5,00\%$		$\mu = 6,00\%$		$\mu = 7,00\%$		$\mu = 8,00\%$		$\mu = 9,00\%$	
σ	CM	BH	CM	BH	CM	BH	CM	BH	CM	BH
15%	0.58	0.52	0.72	0.64	0.87	0.75	1.00	0.85	1.14	0.95
20%	0.44	0.38	0.54	0.47	0.65	0.55	0.75	0.63	0.86	0.70
25%	0.35	0.30	0.43	0.36	0.52	0.43	0.60	0.49	0.68	0.54
30%	0.29	0.24	0.36	0.29	0.43	0.34	0.50	0.39	0.57	0.44
$w = 0.5$	$\mu = 5,00\%$		$\mu = 6,00\%$		$\mu = 7,00\%$		$\mu = 8,00\%$		$\mu = 9,00\%$	
σ	CM	BH	CM	BH	CM	BH	CM	BH	CM	BH
15%	0.56	0.52	0.69	0.64	0.82	0.75	0.95	0.85	1.07	0.95
20%	0.42	0.38	0.52	0.47	0.61	0.55	0.71	0.63	0.80	0.70
25%	0.33	0.30	0.41	0.36	0.49	0.43	0.56	0.49	0.63	0.54
30%	0.27	0.24	0.34	0.29	0.40	0.34	0.46	0.39	0.52	0.44
$w = 0.8$	$\mu = 5,00\%$		$\mu = 6,00\%$		$\mu = 7,00\%$		$\mu = 8,00\%$		$\mu = 9,00\%$	
σ	CM	BH	CM	BH	CM	BH	CM	BH	CM	BH
15%	0.54	0.52	0.66	0.64	0.78	0.75	0.89	0.85	1.00	0.95
20%	0.40	0.38	0.49	0.47	0.58	0.55	0.66	0.63	0.74	0.70
25%	0.31	0.30	0.38	0.36	0.45	0.43	0.52	0.49	0.58	0.54
30%	0.25	0.24	0.31	0.29	0.37	0.34	0.42	0.39	0.47	0.44

Table 5: Sharpe Omega ratio of constant mix and buy-and-hold strategies

$w = 0.2$	$\mu = 5,00\%$		$\mu = 6,00\%$		$\mu = 7,00\%$		$\mu = 8,00\%$		$\mu = 9,00\%$	
σ	CM	BH	CM	BH	CM	BH	CM	BH	CM	BH
15%	32.24	42.45	49.84	66.04	77.61	103	122	164	194	264
20%	12.23	15.18	17	21.01	23.34	29.10	32.26	40.40	44.70	56.26
25%	6.76	8.17	8.82	10.67	11.47	13.88	14.86	18.03	19.25	23.41
30%	4.48	5.35	5.65	6.74	7.08	8.45	8.83	10.56	11	13.17
$w = 0.5$	$\mu = 5,00\%$		$\mu = 6,00\%$		$\mu = 7,00\%$		$\mu = 8,00\%$		$\mu = 9,00\%$	
σ	CM	BH	CM	BH	CM	BH	CM	BH	CM	BH
15%	8.84	9.13	13.66	14.13	21.02	21.76	32.36	33.57	50.06	52.03
20%	4.49	4.63	6.34	6.56	8.86	9.17	12.30	12.74	17.00	17.65
25%	2.89	2.99	3.90	4.04	5.18	5.37	6.80	7.07	8.90	9.25
30%	2.10	2.18	2.75	2.86	3.54	3.69	4.51	4.71	5.70	5.95
$w = 0.8$	$\mu = 5,00\%$		$\mu = 6,00\%$		$\mu = 7,00\%$		$\mu = 8,00\%$		$\mu = 9,00\%$	
σ	CM	BH	CM	BH	CM	BH	CM	BH	CM	BH
15%	6.36	6.41	9.91	9.99	15.30	15.44	23.56	23.78	36.34	36.71
20%	3.44	3.47	4.94	4.98	6.96	7.03	9.70	9.81	13.46	13.62
25%	2.29	2.32	3.15	3.18	4.23	4.28	5.61	5.69	7.37	7.48
30%	1.71	1.73	2.28	2.31	2.97	3.02	3.82	3.88	4.86	4.94

3.5 Optimal portfolio for the basic example (GBM case)

Consider the risky asset dynamics defined by: (see Appendix for the multidimensional case)

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \tag{29}$$

with constant parameters μ and σ ($\sigma > 0$). Denote:

$$\begin{aligned} m &= \mu - \frac{1}{2}\sigma^2;\ \theta = \frac{\mu - r}{\sigma} \text{ (Sharpe ratio)}, \\ a &= -\frac{1}{2}\theta^2 T + \frac{\theta}{\sigma} mT;\ \kappa = \frac{\theta}{\sigma};\ \chi = e^a (S_0)^\kappa. \end{aligned} \tag{30}$$

We consider the risk-neutral probability measure $\mathbb{Q}$ to price options. The $\sigma-$algebra $\mathcal{F}$ is generated by the Brownian motion W. We deduce that the probability density function (pdf) of $\frac{d\mathbb{Q}}{dP}$ is a function $g(S_T)$ of the terminal value of the risky asset price with respect to the σ-algebra generated by W. This function is given by:

$$g(s) = \chi s^{-\kappa}.$$

For the standard decision criterion, the utility U is also supposed to be concave, meaning that the investor is risk averse. Let denote by J the inverse function of the marginal utility, i.e. $J = (U')^{-1}$.

Using Cox and Huang (1989) result, we deduce that the optimal portfolio is equal to:

$$V^*(S_T) = J\left[\lambda\ \chi\ S_T^{-\kappa}\right], \tag{31}$$

where λ is the Lagrange parameter associated to the budget constraint.

Assume for example that the utility function is a power function $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$ with a constant relative risk aversion $\gamma > 0$ and $\gamma \neq 1$. Then, we have: $U'(x) = x^{-\gamma}$and $J(y) = y^{\frac{-1}{\gamma}}$.

For the GBM case, the optimal solution for the CRRA case is given by:

$$V^*(S_T) = c.\chi^{-\frac{1}{\gamma}}.S_T^{\frac{\kappa}{\gamma}}, \tag{32}$$

where the power $\frac{\kappa}{\gamma}$ of S_T is equal to the Sharpe type ratio[13] $\kappa = \frac{\mu - r}{\sigma^2}$multiplied by the inverse of the relative risk aversion γ. The ratio $\frac{\kappa}{\gamma}$ corresponds to the Merton ratio. Applying budget constraint, the coefficient c is equal to:

$$c = \frac{V_0 e^{rT}}{E\left[\left(\chi S_T^{-\kappa}\right)^{\frac{\gamma-1}{\gamma}}\right]} \text{ with } E\left[\left(\chi S_T^{-\kappa}\right)^{\frac{\gamma-1}{\gamma}}\right] = \exp\left(\frac{1}{2}\theta^2 T \frac{1-\gamma}{\gamma^2}\right). \tag{33}$$

Therefore, the optimal portfolio is a power of the terminal risky asset value. Note that $V^*(S_T) =$

[13] We call it "Sharpe type ratio" since it is equal to the Sharpe ratio when we consider the standard deviation instead the variance.

$h^*(S_T)$ is increasing. This property is satisfied for all concave utilities, as soon as the density g is decreasing, for instance within the Black-Scholes asset pricing framework. The concavity/convexity of the optimal payoff is determined by the comparison between the relative risk-aversion γ and the ratio $\kappa = \frac{\mu - r}{\sigma^2}$, which is the Sharpe ratio divided by the volatility σ called the Merton ratio[14]:

i) h^* is concave if $\kappa < \gamma$; ii) h^* is linear if $\kappa = \gamma$; iii) h^* is convex if $\kappa > \gamma$.

Remark 1 *(Condition on weight) The optimal portfolio corresponds to a constant mix strategy with weight $w^* = \frac{\kappa}{\gamma}$. To get condition $0 \leq w^* \leq 1$, the relative risk aversion γ must be higher than κ. To get condition $0 \leq w^* \leq 1$, the relative risk aversion γ must be higher than the ratio $\kappa = \frac{\mu - r}{\sigma^2}$, which is the Sharpe ratio divided by the volatility σ called the Merton ratio.*

3.6 Compensating variation (monetary loss)

Several financial institutions have understood the importance to evaluate the investor's risk aversion although there are difficulties to rigorously link those measures to investment recommendations. To the best of our knowledge, one of the reasons is that those evaluations often do not provide a quantitative evaluation of investor risk aversion. However, Ben-Akiva et al (2002) propose an econometric approach to gauge the investors' risk aversion. Introduced by Hicks (1939) in economics and by De Palma and Prigent (2008, 2009) in finance, the compensating variation allows to quantitatively measure the monetary loss of not receiving his own optimal portfolio. We introduce this notion to compare the two portfolio strategies.

3.6.1 Compensating variation

Consider an investor with utility function parametrized by ζ, time horizon T and initial investment V_0. Assume that he faces the choice between two strategies 1 and 2. Denote by $V_{T,\zeta,1}$ his first portfolio value at maturity and by $V_{T,\zeta,2}$ the second one. Assume that he prefers the first one. Then, we search the initial investment value $\widehat{V}_0$ necessary to reach the same utility level if he chooses strategy 2. Such condition leads to the following indifference condition:

$$E[U_\zeta(V_{T,\zeta,1}; V_0)] = E[U_\zeta(V_{T,\zeta,2}; \widehat{V}_0)]. \tag{34}$$

The ratio $\widehat{V}_0/V_0$, called the compensating variation, provides a quantitative (monetary) measure of the compensation for the less good strategy 2.[15]

[14] See e.g. Prigent (2007).

[15] In De Palma and Prigent (2008), it is shown that the compensating variation can be also related to the certainty equivalent notion.

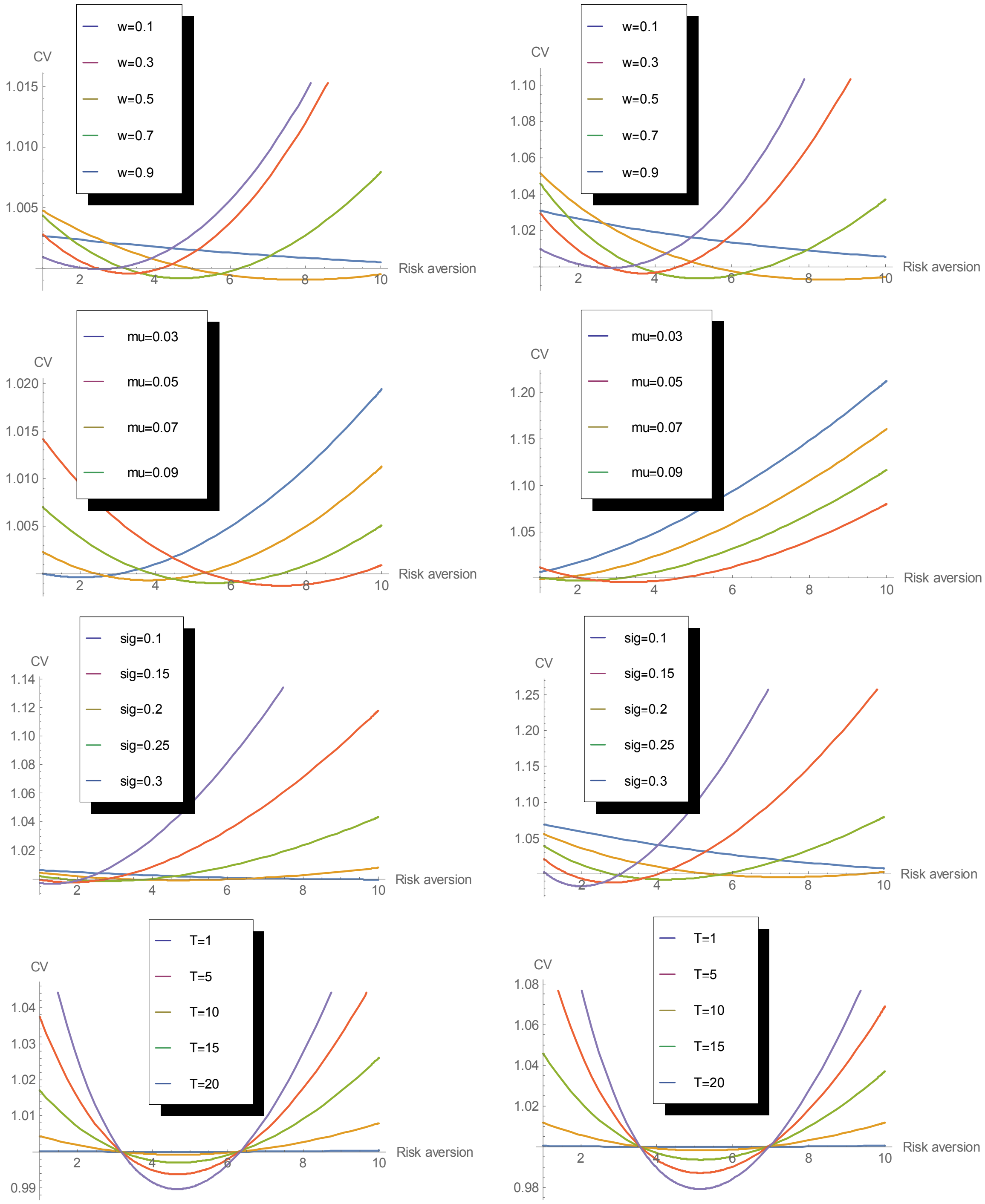
3.6.2 Compensating variation in the GBM framework

In what follows, we provide numerical examples of the compensating variations for various cases for the standard GBM framework. Our two numerical base cases are (1) $\mu = 0.06; r = 0.01; \sigma = 0.15; T = 5$, (2) $\mu = 0.12; r = 0.04; \sigma = 0.18; T = 5$.

We compute the compensating variation for the CRRA case. Note that it encompasses the logarithmic utility case corresponding to the Kelly criterion and the expected growth rate. Using the indifference condition (34), we get:

$$\frac{V_0^{CM}}{V_0^{BH}} = \left(\frac{E\left[\left((1-w)e^{rT} + w \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma W_T\right]\right)^{1-\gamma}\right]}{E\left[\exp\left[\left[(1-w)r + w\mu - \frac{1}{2}w^2\sigma^2\right]T + w\sigma W_T\right]^{1-\gamma}\right]} \right)^{\frac{1}{1-\gamma}}.$$

When the previous ratio is higher than 1, it means that the buy-and-hold is preferable, while, when this ratio is smaller than 1, it means the converse. Since we consider here a time horizon equal to 5 years, we can compare the the compensating variation values to implicit management cost applied on this time period. For example, if the compensating variation is equal to 1.10, we can consider that the investor bears an implicit cost of about 2% per year if not having her optimal portfolio weight. Indeed, note that, for the CRRA case in the GBM framework, for each weight w, there exists a relative risk aversion γ such that the constant mix strategy corresponding to the given weight w is optimal. Indeed, as mentioned previously, we have: $w = \frac{1}{\gamma}\frac{\mu - r}{\sigma^2}$. Thus, in that case, we have to compensate the Buy-and-Hold strategy to get the same expected utility level, implying that the ratio $\frac{V_0^{CM}}{V_0^{BH}}$ is smaller than one. Note that, if we consider another utility function, for example a CARA utility defined by $U(x) = -e^{-ax}/a$ where a corresponds to the constant absolute risk aversion, then the constant mix is never optimal (see Appendix for the compensating variation in the CARA case). As illustrated by Figure (7), when the constant mix is preferable, the compensating variation of the buy-and-hold is weak, whereas, when the buy-and-hold, the compensating variation of the constant mix can be very high. It means that, when buy-and-hold strategy outperforms rebalancing one with respect to an utility function, it is far more significantly. Indeed, when the constant mix is preferable, the minimum compensating variation is around 0.98 (which means approximately that the investor bears an implicit cost equal to 0.4% per year). When the buy-and-hold strategy is preferable, the maximum compensating variation can reach 20% (approximately, 4% per year). We can check also that when the weight is close to 0 or 1, the compensating variation is close to 1. Indeed, for very small or very high values of the weight w, the rebalancing and the buy-and-hold strategies are close. The higher the trend, the higher the compensating variation when the buy-and-hold strategy is preferable. The higher the volatility, the higher the compensating variation when the buy-and-hold strategy is preferable. Finally, as function of the time horizon, the compensating variation is increasing. For small time horizon ($T = 1$ year), it is quite close to 1, meaning that there is almost no compensation. Indeed, in that case, the two strategies are very close.



Case 1 Case 2

Figure 7: Compensating variations.

4 Empirical analysis

In what follows, our objective is to compare Buy-and-Hold (BH) and Constant Mix (CM) strategies for two financial indices representative of the stock market on one side and on a safer asset on the other side. For this purpose, using actual data, we rely on the stationary block bootstrap method of Politis and Romano (1994). In order to estimate the optimal block length of block bootstrap methods for dependent data, we use the automatic block-length of Politis and White (2004) and more precisely the correction made by Patton et al. (2009). We compare the two strategies using US monthly data that cover the sample period from July 1963 to December 2019. The US stock market is represented by the S&P 500 Total return index (dividends included). From the 10-year Treasury constant maturity rate time-series, we approximate long term bond total return using the usual loglinear approximation formula described in chapter 10 of Campbell, Lo and MacKinlay (1997). The 1-month T-Bill return is from Ibbotson and Associates, Inc. In Table 6, we present the statistics for US monthly returns over the whole sample period while, in Table 7, we provide the correlations within the US financial assets.

Table 6: Statistics for US Market Returns (1963:07 - 2019:12)

	SP500	T-Bill	Bond
Mean	0.87%	0.38%	0.56%
Volatility	3.49%	0.26%	1.96%
Skewness	-0.752	0.635	0.789
Kurtosis	6.206	3.720	7.103
Min	-20.19%	0.00%	-7.64%
Max	12.32%	1.35%	11.39%
p-value JB Test	0.001	0.001	0.001

Notes: This table presents the main descriptive statistics of the monthly returns of S&P 500 and US Bond and T-Bill from from July 1963 to December 2019.

Table 7: Correlation US Market (1963:07 - 2019:12)

	T-Bill	**Bond**	**S&P 500**
T-Bill	1.000	0.015	-0.024
Bond	0.015	1.000	0.088
S&P 500	-0.024	0.088	1.000

Notes: This table presents the correlations within the US financial assets from July 1963 to December 2019.

For the bootstrap analysis, we perform 10^6 resamples of 60, 120 and 240 months BH and CM portfolio strategies. Table 8 (resp. 9) displays various statistics for the BH and CM strategies for S&P 500 and T-Bill assets (resp. for S&P 500 and Bond).

Table 8: BH and CM Statistics SP 500/T-Bill Portfolios

Panel A	**60 months**											
	w(SP 500)											
	0.010	0.027	0.126	0.226	0.326	0.425	0.525	0.625	0.724	0.824	0.924	0.990
P(CM>BH)	0.423	0.424	0.425	0.427	0.429	0.431	0.433	0.435	0.437	0.439	0.441	0.442
Mean(CM)	0.261	0.267	0.304	0.343	0.384	0.426	0.469	0.515	0.562	0.612	0.663	0.699
Mean(BH)	0.262	0.269	0.314	0.358	0.403	0.447	0.492	0.537	0.581	0.626	0.670	0.700
Std(CM)	0.072	0.072	0.085	0.115	0.155	0.201	0.251	0.305	0.363	0.425	0.492	0.538
Std(BH)	0.071	0.072	0.094	0.136	0.185	0.237	0.289	0.342	0.396	0.450	0.504	0.540
SR(CM)	0.051	0.136	0.553	0.743	0.813	0.837	0.845	0.844	0.840	0.834	0.826	0.820
SR(BH)	0.063	0.166	0.599	0.740	0.786	0.803	0.811	0.815	0.817	0.818	0.819	0.819
Skew.(CM)	0.417	0.403	0.222	0.106	0.107	0.159	0.232	0.314	0.401	0.491	0.583	0.646
Skew.(BH)	0.415	0.394	0.403	0.538	0.606	0.635	0.647	0.653	0.655	0.656	0.656	0.656
Kurto.(CM)	3.587	3.540	3.237	3.167	3.155	3.161	3.195	3.260	3.358	3.491	3.659	3.791
Kurto.(BH)	3.580	3.507	3.386	3.660	3.781	3.820	3.831	3.831	3.827	3.822	3.817	3.813
Panel B	**120 months**											
	w(SP 500)											
	0.010	0.027	0.126	0.226	0.326	0.425	0.525	0.625	0.724	0.824	0.924	0.990
P(CM>BH)	0.342	0.343	0.346	0.349	0.353	0.356	0.359	0.363	0.366	0.369	0.373	0.375
Mean(CM)	0.590	0.606	0.701	0.804	0.914	1.032	1.158	1.294	1.440	1.597	1.765	1.884
Mean(BH)	0.594	0.616	0.748	0.880	1.011	1.143	1.275	1.407	1.538	1.670	1.802	1.889
Std(CM)	0.130	0.131	0.159	0.222	0.306	0.409	0.527	0.662	0.815	0.988	1.184	1.328
Std(BH)	0.130	0.133	0.208	0.324	0.451	0.581	0.714	0.847	0.980	1.114	1.248	1.338
SR(CM)	0.070	0.187	0.755	1.005	1.085	1.102	1.095	1.077	1.053	1.027	1.000	0.981
SR(BH)	0.102	0.265	0.804	0.922	0.955	0.967	0.972	0.975	0.977	0.977	0.978	0.978
Skew.(CM)	0.429	0.419	0.310	0.267	0.319	0.416	0.532	0.660	0.794	0.935	1.083	1.185
Skew.(BH)	0.425	0.408	0.775	1.045	1.137	1.173	1.188	1.195	1.198	1.200	1.200	1.200
Kurto.(CM)	3.486	3.461	3.255	3.193	3.232	3.343	3.526	3.785	4.125	4.556	5.089	5.509
Kurto.(BH)	3.474	3.414	4.327	5.142	5.414	5.513	5.553	5.570	5.576	5.578	5.578	5.577
Panel C	**240 months**											
	w(SP 500)											
	0.010	0.027	0.126	0.226	0.326	0.425	0.525	0.625	0.724	0.824	0.924	0.990
P(CM>BH)	0.216	0.216	0.220	0.225	0.229	0.233	0.238	0.242	0.247	0.252	0.257	0.260
Mean(CM)	1.529	1.578	1.894	2.254	2.663	3.129	3.660	4.265	4.957	5.748	6.653	7.328
Mean(BH)	1.559	1.658	2.249	2.841	3.432	4.024	4.615	5.207	5.798	6.390	6.981	7.376
Std(CM)	0.296	0.300	0.386	0.569	0.836	1.188	1.634	2.194	2.892	3.761	4.841	5.702
Std(BH)	0.299	0.329	0.787	1.345	1.917	2.494	3.073	3.653	4.234	4.815	5.396	5.784
SR(CM)	0.098	0.260	1.022	1.325	1.391	1.372	1.322	1.261	1.195	1.129	1.065	1.022
SR(BH)	0.199	0.479	0.952	0.997	1.008	1.012	1.014	1.015	1.015	1.016	1.016	1.016
Skew.(CM)	0.480	0.473	0.423	0.454	0.571	0.732	0.918	1.122	1.345	1.587	1.852	2.045
Skew.(BH)	0.473	0.566	1.763	1.985	2.040	2.059	2.067	2.071	2.073	2.074	2.075	2.075
Kurto.(CM)	3.493	3.473	3.359	3.394	3.593	3.954	4.495	5.248	6.262	7.611	9.398	10.905
Kurto.(BH)	3.466	3.741	9.486	10.682	10.976	11.078	11.120	11.140	11.150	11.155	11.157	11.158

Table 9: BH and CM Statistics SP 500/Bond Portfolios

Panel A	60 months											
	w(SP 500)											
	0.010	0.027	0.126	0.226	0.326	0.425	0.525	0.625	0.724	0.824	0.924	0.990
P(CM>BH)	0.511	0.512	0.513	0.515	0.516	0.518	0.519	0.521	0.522	0.524	0.526	0.527
Mean(CM)	0.408	0.412	0.437	0.463	0.490	0.518	0.548	0.578	0.610	0.642	0.676	0.700
Mean(BH)	0.408	0.413	0.443	0.473	0.502	0.532	0.562	0.591	0.621	0.651	0.680	0.700
Std(CM)	0.256	0.253	0.244	0.244	0.256	0.278	0.309	0.348	0.393	0.444	0.500	0.540
Std(BH)	0.255	0.253	0.243	0.247	0.263	0.290	0.325	0.366	0.411	0.458	0.508	0.541
SR(CM)	0.588	0.610	0.737	0.842	0.910	0.940	0.940	0.923	0.896	0.867	0.838	0.819
SR(BH)	0.591	0.617	0.764	0.873	0.931	0.947	0.936	0.913	0.886	0.859	0.834	0.818
Skew.(CM)	0.889	0.895	0.924	0.916	0.850	0.749	0.655	0.594	0.571	0.582	0.618	0.653
Skew.(BH)	0.893	0.904	0.930	0.901	0.852	0.807	0.773	0.744	0.718	0.695	0.674	0.661
Kurto.(CM)	4.564	4.583	4.672	4.642	4.460	4.206	3.972	3.807	3.719	3.704	3.752	3.816
Kurto.(BH)	4.573	4.604	4.687	4.588	4.406	4.239	4.114	4.024	3.956	3.902	3.857	3.832
Panel B	**120 months**											
	w(SP 500)											
	0.010	0.027	0.126	0.226	0.326	0.425	0.525	0.625	0.724	0.824	0.924	0.990
P(CM>BH)	0.475	0.476	0.479	0.482	0.484	0.487	0.490	0.493	0.496	0.499	0.502	0.504
Mean(CM)	0.982	0.993	1.065	1.141	1.221	1.306	1.396	1.491	1.592	1.698	1.811	1.890
Mean(BH)	0.984	0.999	1.092	1.184	1.277	1.369	1.461	1.554	1.646	1.738	1.831	1.892
Std(CM)	0.513	0.510	0.499	0.509	0.543	0.601	0.682	0.784	0.907	1.048	1.210	1.328
Std(BH)	0.512	0.507	0.500	0.529	0.591	0.675	0.775	0.885	1.002	1.124	1.249	1.334
SR(CM)	0.781	0.809	0.971	1.100	1.179	1.207	1.195	1.160	1.115	1.066	1.017	0.985
SR(BH)	0.787	0.825	1.022	1.140	1.178	1.168	1.136	1.099	1.063	1.030	1.001	0.983
Skew.(CM)	1.060	1.059	1.055	1.039	0.996	0.943	0.907	0.906	0.942	1.009	1.102	1.175
Skew.(BH)	1.066	1.072	1.051	1.029	1.058	1.111	1.156	1.183	1.195	1.197	1.192	1.188
Kurto.(CM)	5.196	5.189	5.163	5.084	4.922	4.733	4.599	4.566	4.649	4.851	5.176	5.463
Kurto.(BH)	5.216	5.232	5.142	5.008	5.079	5.256	5.408	5.498	5.537	5.542	5.529	5.514
Panel C	**240 months**											
	w(SP 500)											
	0.010	0.027	0.126	0.226	0.326	0.425	0.525	0.625	0.724	0.824	0.924	0.990
P(CM>BH)	0.406	0.407	0.412	0.417	0.421	0.426	0.431	0.436	0.442	0.447	0.452	0.456
Mean(CM)	2.929	2.975	3.266	3.585	3.935	4.320	4.743	5.208	5.720	6.284	6.906	7.356
Mean(BH)	2.947	3.022	3.473	3.924	4.375	4.826	5.277	5.728	6.179	6.630	7.081	7.382
Std(CM)	1.466	1.464	1.483	1.568	1.737	2.000	2.366	2.841	3.436	4.165	5.047	5.732
Std(BH)	1.461	1.454	1.549	1.835	2.240	2.711	3.220	3.751	4.295	4.849	5.410	5.786
SR(CM)	0.974	1.007	1.191	1.329	1.402	1.410	1.371	1.305	1.228	1.149	1.071	1.022
SR(BH)	0.990	1.046	1.273	1.321	1.283	1.227	1.173	1.127	1.089	1.058	1.032	1.017
Skew.(CM)	1.397	1.387	1.350	1.328	1.309	1.305	1.336	1.411	1.533	1.699	1.908	2.071
Skew.(BH)	1.405	1.400	1.342	1.499	1.727	1.893	1.992	2.047	2.076	2.091	2.096	2.097
Kurto.(CM)	6.875	6.824	6.622	6.486	6.358	6.299	6.411	6.770	7.438	8.483	10.018	11.402
Kurto.(BH)	6.916	6.890	6.551	7.539	9.081	10.215	10.891	11.268	11.472	11.577	11.626	11.640

For all portfolio weights (from 1% to 99% in the S&P 500), the mean return of CM strategy is below the one of the BH strategy. The same is true for the volatility. Note also that the probability that the CM strategy ends up with a higher portfolio value after 20 years than the BH strategy, is between 11.3% and 14.4%, a very low value. We also compute the Sharpe ratio. This statistics is clearly in favor of the CM strategy. We also consider higher moments of strategy returns: skewness and kurtosis. The BH strategy exhibits higher level of skewness and kurtosis than the CM strategy. Note that in this case a high level of kurtosis associated with a high value for the skewness is not necessarily a bad thing since the extreme returns are essentially positive. Such results about comparison of the first four moments are quite similar to those proved by Wise (2006) in the geometric Brownian motion framework and to our results in Section 3 for more general diffusion processes.

To visualize how the shape of the distribution of returns of both strategies matters, we plot their probability distribution functions (pdf) in Figures 8 and 9 for a time horizon of 120 months[16]. This allows to illustrate in particular why the buy-and-hold strategy exhibits much higher level of skewness and kurtosis than the constant mix strategy.

[16] In Appendix, we provide additional empirical results about the pdf of both strategies for both 60 and 120 months.

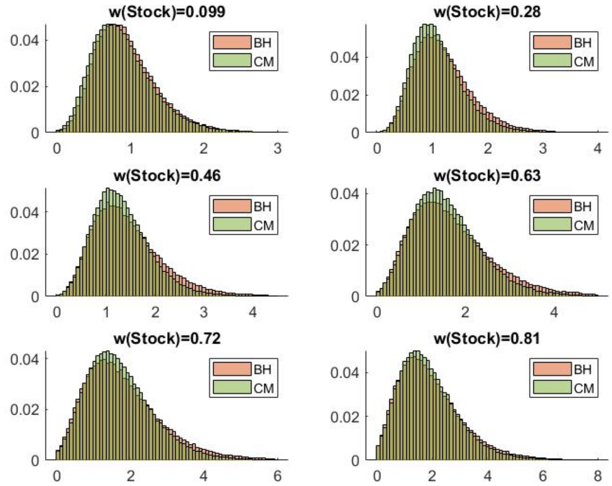


Figure 8: Pdf of CM and BH Portfolios US Stock T-Bill (120 months)

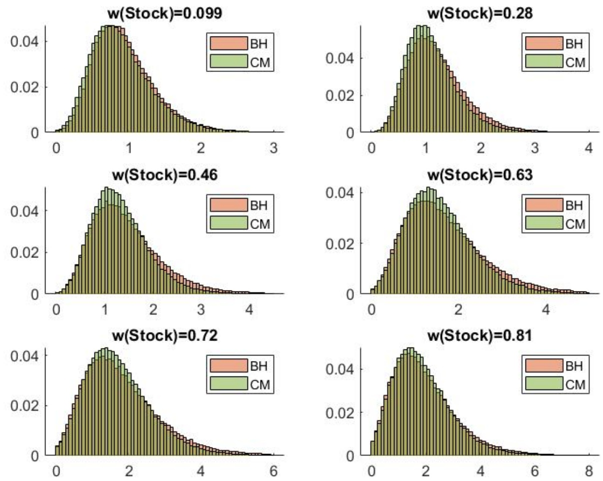


Figure 9: Pdf of CM and BH Portfolios US Stock Bond (120 months)

Based on all previous metrics analyzed so far, it is not straightforward to say which strategies will be the best for investors. Of course, this comparison depends on the chosen comparison criterion. If we examine the cumulative distributions functions (cdf), we find that there is no stochastic dominance at the second order and the two cdf curves are rather close, especially for short time horizons.

Therefore, to better take account of the whole distribution of returns as well as investors risk aversion, we examine now the compensating variation (CV) approach. To get condition $0 \leq w^* \leq 1$, the relative risk aversion γ must be higher than the Merton ratio (see Remark 1). But, as shown in Section 3, even for this favourable case, the constant mix is not very significantly dominant with respect to the compensating variation. Note also that, if we consider another utility function, for example a CARA utility defined by $U(v) = -e^{-av}/a$ where a corresponds to the constant absolute risk aversion, then the constant mix is never optimal.

Figure 10 displays the CV as a function of the weights invested in the risky asset (S&P 500) and of the risk aversion parameter γ. As deduced from the theoretical model, we can see that the BH strategy dominates the CM in the CV sense most of the time (*i.e.*, blue area on figures). Note also that, when the constant mix is preferable, the compensating variation of the buy-and-hold is weak, whereas, when the buy-and-hold is preferable, the compensating variation of the constant mix can be very high. It means that, when buy-and-hold strategy outperforms rebalancing one with respect to an utility function, it is far more significantly. Indeed, for example for a ten years investment period, when the constant mix is preferable, the minimum compensating variation is around 0.98 (which means approximately that the investor bears an implicit cost equal to 0.4% per year). When the buy-and-hold strategy is preferable, the maximum compensating variation can reach 80% (approximately, 10% per year).

We can check also that when the weight is close to 0 or 1, the compensating variation is close to 1. Indeed, for very small or very high values of the weight w, the rebalancing and the buy-and-hold strategies are obviously close.[17]

[17] The results for the French market are qualitatively the same as those of the US market see Appendix).

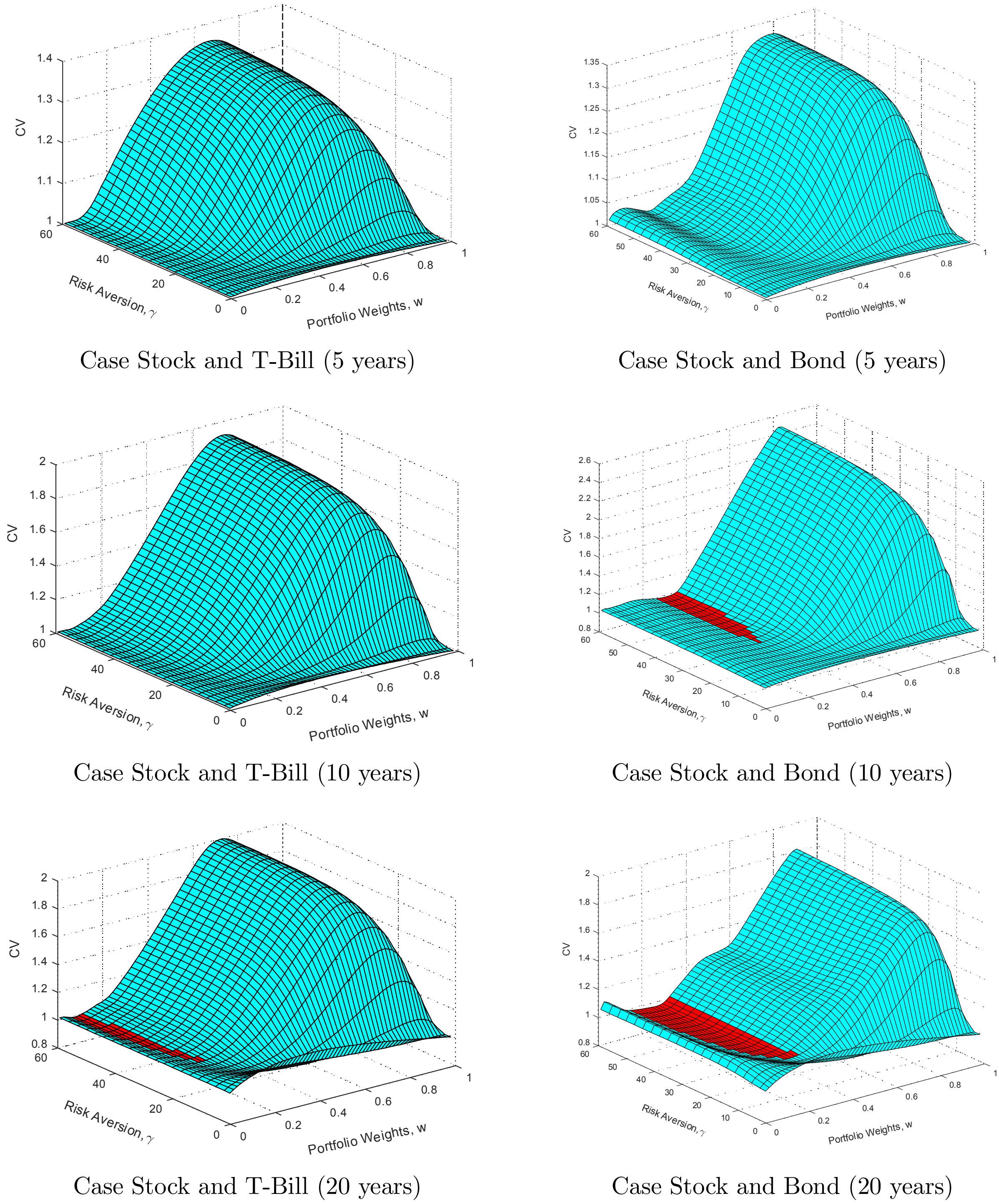


Figure 10: Compensating variations of the BH strategy versus the CM one. *Note*: When the CV is higher than 1, it means that the BH strategy dominates the CM one. The color of the area where the BH strategy dominates the CM in the CV sense is blue (red area when it is the converse)

5 Conclusion

We have examined and compared the rebalancing (constant mix) and buy-and-hold portfolio strategies. We have considered various criteria such as comparison of payoffs, of return cumulative distribution functions and performance measures such as the Sharpe ratio and Kappa measures. We have also introduced the notion of compensating variation to gauge their respective expected utilities. Our study reveals that, even if the probability that the constant mix payoff is generally higher than the buy-and-hold payoff (often around 66%, at least in the GBM framework), this superiority is not very significant. Indeed, for example, when the constant mix is preferable, the compensating variation of the buy-and-hold is weak, whereas, when the buy-and-hold is preferable, the compensating variation of the constant mix can be very high. Therefore, when buy-and-hold strategy outperforms rebalancing one with respect to an utility function, it is far more significantly. These results are confirmed by the empirical study of the two strategies based on US monthly data that cover the sample period from July 1963 to December 2019.

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6 Appendix

In this Appendix, we detail proofs of main comparison results.

6.1 Proof of the intersection of payoffs and computation of the maximal rebalancing return

We have to examine the following equation: find $R \geq 0$ such that

$$\lambda R^w = wR + (1-w)e^{rT},$$

where $\lambda = \exp\left[\left[(1-w)r + \frac{\sigma^2}{2}\left(w - w^2\right)\right]T\right]$ is strictly positive.

Introduce the function φ defined by:

$$\varphi(R) = \lambda R^w - wR - (1-w)e^{rT}.$$

We get:

$$\varphi'(R) = \lambda w R^{w-1} - w,$$

implying that:

$$\varphi'(R^*) = 0 \Longleftrightarrow R^* = \left(\frac{1}{\lambda}\right)^{\frac{1}{w-1}}.$$

Since we have $\varphi'(R^*) = \lambda w R^{*w-1} - w = 0$, we deduce that $\lambda R^{*w} = R^*$ thus

$$\varphi(R^*) = (1-w)\left[R^* - e^{rT}\right].$$

<u>Case $0 < w < 1$</u> (Long-only)

In that case, we get:

$$\varphi(R^*) > 0 \Longleftrightarrow R^* \geq e^{rT} \Longleftrightarrow \left(\frac{1}{\lambda}\right)^{\frac{1}{w-1}} > e^{rT}.$$

Recall that $\lambda = \exp\left[\left[(1-w)r + \frac{\sigma^2}{2}\left(w - w^2\right)\right]T\right]$. Thus:

$$\begin{aligned}\left(\frac{1}{\lambda}\right)^{\frac{1}{w-1}} &= \exp\left[-\frac{1}{w-1}\left[(1-w)r + \frac{\sigma^2}{2}\left(w - w^2\right)\right]T\right] \\ &= \exp\left[rT + w\frac{\sigma^2}{2}T\right].\end{aligned}$$

Consequently we get $\left(\frac{1}{\lambda}\right)^{\frac{1}{w-1}} > e^{rT}$ which implies that $\varphi(R^*) > 0$. Now, we note that $\varphi(0) = -(1-w)e^{rT} < 0$ and $\lim_{R\to\infty} \varphi(R) = -\infty$. Finally, using intermediate value theorem for continuous functions jointly with strictly monotony of function φ on both subintervals $[0, R^*]$ and $[R^*, +\infty[$, we deduce that there exist exactly two values of the risky asset return such that the rebalancing return R^R is null, meaning that the two payoffs intersect. We note also that the rebalancing return is R^R is maximal at $R^* = \exp\left[rT + w\frac{\sigma^2}{2}T\right]$.[18]

6.2 Impact of jumps

Recall that, when the risky asset dynamics is a pure jump process, we get:

$$R_T^D =$$

$$\exp\left[(1-w)rT\right]\prod_{T_n \leq T}\left(1 + w\delta(T_n, S_{T_n-})\right) - (1-w)e^{rT} - w\prod_{T_n \leq T}\left(1 + \delta(T_n, S_{T_n-})\right).$$

For $r = 0$, we have to compare $\prod_{T_n \leq T}\left(1 + w\delta(T_n, S_{T_n-})\right)$ with $(1-w) + w\prod_{T_n \leq T}\left(1 + \delta(T_n, S_{T_n-})\right)$.

We have:

$$R_T^D \geq 0 \Longleftrightarrow$$

$$\prod_{T_n \leq T}\left(1 + w\delta(T_n, S_{T_n-})\right) \geq (1-w) + w\prod_{T_n \leq T}\left(1 + \delta(T_n, S_{T_n-})\right). \tag{35}$$

First case: Assume that all relative jumps are negative. Then inequality (35) is equivalent to:

$$\frac{\prod_{T_n \leq T}\left(1 + w\delta(T_n, S_{T_n-})\right) - 1}{\prod_{T_n \leq T}\left(1 + \delta(T_n, S_{T_n-})\right) - 1} \leq w.$$

Then, when all relative jumps are negative, we get $\left(1 + w\delta(T_n, S_{T_n-})\right) \geq \left(1 + \delta(T_n, S_{T_n-})\right)$ for all n, implying that $\frac{\prod_{T_n \leq T}(1+w\delta(T_n,S_{T_n-}))-1}{\prod_{T_n \leq T}(1+\delta(T_n,S_{T_n-}))-1} \geq 1 \geq w$ since $0 \leq w \leq 1$.

It means that $R_T^D \leq 0$ or equivalently $R_T^{CM} \leq R_T^{BH}$.

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[18] Even if we consider the long only case, we can prove that we get similar results for the two other cases:

Case $w < 0$ (short, inverse leveraged)

In that case, we get $\varphi(R^*) < 0$ and also $\lim_{R \to 0^+}\varphi(R) = +\infty$ and $\lim_{R \to \infty}\varphi(R) = +\infty$. Thus we deduce also that there exist exactly two return values of the risky asset at which the two payoffs intersect. We note also that the rebalancing return is R^R is minimal at $R^* = \exp\left[rT + w\frac{\sigma^2}{2}T\right]$.

Case $w > 1$ (leveraged)

In that case, we get $\varphi(R^*) < 0$ and $\varphi(0) = -(1-w)e^{rT} > 0$ and $\lim_{R \to \infty}\varphi(R) = +\infty$. Thus we deduce also that there exist exactly two return values of the risky asset at which the two payoffs intersect.

We note also that the diversification return is R^D is minimal at $R^* = \exp\left[rT + w\frac{\sigma^2}{2}T\right]$.

Second case: Assume that all relative jumps are positive. Then inequality (35) is equivalent to:

$$\frac{\prod_{T_n \le T} (1 + w\delta(T_n, S_{T_n-})) - 1}{\prod_{T_n \le T} (1 + \delta(T_n, S_{T_n-})) - 1} \ge w.$$

Then, when all relative jumps are positive, we get $(1 + w\delta(T_n, S_{T_n-})) \le (1 + \delta(T_n, S_{T_n-}))$ for all n, implying that $\frac{\prod_{T_n \le T} (1+w\delta(T_n,S_{T_n-}))-1}{\prod_{T_n \le T} (1+\delta(T_n,S_{T_n-}))-1} \le 1$. To examine the sign of the rebalancing return, we use the following lemma:

Lemma 2 *Let $(a_n)_{n\ge1}$ be a sequence of real numbers with $a_n \ge 1$for all n. Let $0 \le w \le 1$. Then, we have:*

$$\prod_{n \le N} ((1 - w) + wa_n) \le (1 - w) + w \prod_{n \le N} a_n.$$

Proof. (By recurrence)

- For $N = 2$, we have:

$$\begin{aligned} & (1 - w) + wa_1a_2 - ((1 - w) + wa_1) ((1 - w) + wa_2) \\ = \; & (1 - w) + wa_1a_2 - (1 - w)^2 - w (1 - w) (a_1 + a_2) - w^2 a_1a_2 \\ = \; & (1 - w) w \times [1 + a_1a_2 - (a_1 + a_2)] . \end{aligned}$$

Now consider the function $\varphi_{a_2}(x) = 1 + a_2x - (x + a_2) = 1 - a_2 + (a_2 - 1) x$ on $[1, +\infty[$. Since $a_2 - 1 \ge 1$, $\varphi_{a_2}(x)$ is increasing on $[1, +\infty[$ with $\varphi_{a_2}(1) = 0$. Thus $\varphi_{a_2}(x) \ge 0$ on $[1, +\infty[$. Therefore, $\varphi_{a_2}(a_1) \ge 0$ which is equivalent to:

$$(1 - w) + wa_1a_2 \ge ((1 - w) + wa_1) ((1 - w) + wa_2) .$$

- Assume that

$$\prod_{n \le N} ((1 - w) + wa_n) \le (1 - w) + w \prod_{n \le N} a_n.$$

Then, we get:

$$\prod_{n \le N+1} ((1 - w) + wa_n) \le ((1 - w) + wa_{N+1}) \left[(1 - w) + w \prod_{n \le N} a_n\right].$$

Since both a_{N+1} and $\prod_{n \le N} a_n$ are higher than 1, we can apply the property which is true for $N = 2$

and we get the result:

$$\prod_{n\leq N+1}((1-w)+wa_n)\leq\left[(1-w)+w\prod_{n\leq N+1}a_n\right].$$

Since $1+w\delta(T_n,S_{T_n-})=(1-w)+w\,(1+\delta(T_n,S_{T_n-}))$, applying previous lemma with $a_n=1+\delta(T_n,S_{T_n-})$, we deduce that $R_T^D\leq 0$ or equivalently $R_T^{CM}\leq R_T^{BH}$. ■

Third case: (one example of CM dominance with jumps)

Assume that there exist two periods with one period corresponding to a rise and one period corresponding to a drop. For the BH strategy, we get a return equal to $(1-w)+wab$. For the CM strategy, we get a return equal to $((1-w)+wa)\,((1-w)+wb)$ with $a>1$and $0<b<1$. Consider the function $\varphi_b(x)=1+bx-(x+b)=1-b+(b-1)\,x$ on $[1,+\infty[$. Since $0<b<1$, $\varphi_b(x)$ is decreasing on $[1,+\infty[$ with $\varphi_{a_2}(1)=0$. Therefore, $\varphi_b(a)\leq 0$ which is equivalent to:

$$(1-w)+wab\leq((1-w)+wa)\,((1-w)+wb)\,.$$

We deduce that $R_T^D\geq 0$ or equivalently $R_T^{CM}\geq R_T^{BH}$.

6.3 Comparison of four moments

- In what follows, we begin by proving Relations (13) and (14). To compute the return expectations of both strategies, we use the following lemma:

Lemma 3 *Suppose that the dynamics of process X satisfies:*

$$\frac{dX_t}{X_t}=a\,(t)\,dt+b\,(t)\,dW_t+c_t dN_t, \tag{36}$$

where W denotes a standard Brownian motion with respect to a given filtration $(\mathcal{F}_t)_t$, where both the drift $a(.)$ and the volatility $b(.)$ are deterministic and N is a compound Poisson process with intensity λ and $\overline{c}$ the common expectation of the relative jumps of the risky asset $\frac{\Delta X_{T_n}}{X_{T_n-}}=c(T_n,S_{T_n-})$.at jump times T_n. Then we get:

$$E\left[X_T\right]=X_0\exp\left[\int_0^T a(t)dt+\overline{c}\lambda T\right].$$

Proof. *Using Ito's lemma, we deduce that:*

$$X_T=X_0\exp\left[\int_0^T\left(a(t)-\frac{1}{2}b^2\,(t)\right)dt+\int_0^T b\,(t)\,dW_t\right]\prod_{T_n\leq T}(1+c(T_n,S_{T_n-}))\,. \tag{37}$$

Then, by independence of the diffusion and the jump components, we get:

$$E\left[X_T\right]=$$

$$X_0 E\left[\exp\left[\int_0^T \left(a(t)-\frac{1}{2}b^2(t)\right)dt+\int_0^T b(t)\,dW_t\right]\times E\left[\prod_{T_n\le T}(1+c(T_n,S_{T_n-}))\right]\right]=$$

$$X_0\exp\left[\int_0^T a(t)dt\right]\times E\left[\prod_{T_n\le T}(1+c(T_n,S_{T_n-}))\right].$$

To compute $E\left[\prod_{T_n\le T}(1+c(T_n,S_{T_n-}))\right]$*, we use the following decomposition:*

$$\begin{aligned} E\left[\prod_{T_n\le T}(1+c(T_n,S_{T_n-}))\right] &= \sum_{n=0}^{\infty} E\left[\prod_{k=0}^{n}(1+c(T_k,S_{T_k-}))\,|\mathcal{N}_t=n\right]P\left[\mathcal{N}_t=n\right] \\ &= \exp(-\lambda T)\sum_{n=0}^{\infty}(1+\overline{c})^n\frac{(\lambda T)^n}{n!} \\ &= \exp(-\lambda T)\exp((1+\overline{c})\lambda T) \\ &= \exp(\overline{c}\lambda T). \end{aligned}$$

■

Therefore, applying previous lemma when $X_t = V_t^{BH}$ or when $X_t = V_t^{CM}$, we get respectively:

$$E\left[R_T^{BH}\right]=(1-w)e^{rT}+w\exp\left[\int_0^T \mu(t)dt+\overline{\delta}\lambda T\right],$$

$$E\left[R_T^{CM}\right]=\exp\left[(1-w)rT+w\left(\int_0^T \mu(t)dt+\overline{\delta}\lambda T\right)\right]. \quad (38)$$

- To compute the return variances of both strategies, we use the following lemma:

Lemma 4 *Suppose that the dynamics of process X satisfies:*

$$\frac{dX_t}{X_t}=a(t)\,dt+b(t)\,dW_t+c_t dN_t, \quad (39)$$

with same assumptions as previously. Denote by $\overline{c^2}$ the common expectation of the squares of the relative jumps of the risky asset $\frac{\Delta X_{T_n}}{X_{T_n-}}=c(T_n,X_{T_n-})$.at jump times T_n. Then we get:

$$Variance\left[X_T\right]=$$

$$X_0^2\exp\left[2\left(\int_0^T a(t)dt+\overline{c}\lambda T\right)\right][e^{\int_0^T b^2(t)dt+\overline{c^2}\lambda T}-1].$$

Proof. We have:

$$Variance\left[X_T\right]=E\left[X_T^2\right]-E^2\left[X_T\right].$$

We note that:

1)

$$E^2\left[X_T\right] = X_0^2 \exp\left[2\int_0^T a(t)dt\right] \times \exp\left(2\overline{c}\lambda T\right).$$

2)

$$E\left[X_T^2\right] =$$

$$\begin{aligned}
&= X_0^2 E\left[\exp\left[2\int_0^T \left(a(t)-\frac{1}{2}b^2(t)\right)dt + 2\int_0^T b(t)\,dW_t\right] \times \prod_{T_n \leq T}(1+c(T_n,S_{T_n-}))^2\right] \\
&= X_0^2 \exp\left[2\int_0^T a(t)dt\right] \exp\left[\int_0^T b^2(t)dt\right] \times E\left[\prod_{T_n \leq T}(1+c(T_n,S_{T_n-}))^2\right].
\end{aligned}$$

Thus we must compute $E\left[\prod_{T_n \leq T}(1+c(T_n,S_{T_n-}))^2\right]$. We have:

$$\begin{aligned}
E\left[\prod_{T_n \leq T}(1+c(T_n,S_{T_n-}))^2\right] &= \sum_{n=0}^{\infty} E\left[\prod_{k=0}^{n}(1+c(T_k,S_{T_k-}))^2 \,|\mathcal{N}_t = n\right] P\left[\mathcal{N}_t = n\right] \\
&= \exp\left(-\lambda T\right)\sum_{n=0}^{\infty}\left[E\left(1+c\right)^2\right]^n \frac{(\lambda T)^n}{n!} \\
&= \exp\left(-\lambda T\right)\exp\left(\left(1+2\overline{c}+\overline{c^2}\right)\lambda T\right) \\
&= \exp\left(\left(2\overline{c}+\overline{c^2}\right)\lambda T\right).
\end{aligned}$$

Applying previous results, we get:

$$Variance\left[X_T\right] =$$

$$X_0^2 \exp\left[2\left(\int_0^T a(t)dt + \overline{c}\lambda T\right)\right][e^{\int_0^T b^2(t)dt+\overline{c^2}\lambda T} - 1].$$

Thus, we deduce:

$$Variance\left[R_T^{BH}\right] = w^2 \exp\left[2\left(\int_0^T \mu(t)dt + \overline{\delta}\lambda T\right)\right]\left(e^{\int_0^T \sigma^2(t)dt+\overline{\delta^2}\lambda T} - 1\right), \tag{40}$$

$$Variance\left[R_T^{CM}\right] = \exp\left[2\left((1-w)rT + w\left(\int_0^T \mu(t)dt + \overline{\delta}\lambda T\right)\right)\right]\left(e^{w^2\left(\int_0^T \sigma^2(t)dt+\overline{\delta^2}\lambda T\right)} - 1\right). \tag{41}$$

To compare the two variances, denote $m_T = \int_0^T \mu(t)dt + \overline{\delta}\lambda T$ and $s_T^2 = \int_0^T \sigma^2(t)dt + \overline{\delta^2}\lambda T$. We have:

$$Variance\left[R_T^{BH}\right] = w^2 \exp\left[2m_T\right]\left(e^{s_T^2} - 1\right), \tag{42}$$

$$Variance\left[R_T^{CM}\right]=\exp\left[2\left((1-w)rT+w\ m_T\right)\right]\left(e^{w^2s_T^2}-1\right). \tag{43}$$

1) First we note that:

$$\exp\left[2m_T\right]\geq\exp\left[2\left((1-w)rT+w\ m_T\right)\right].$$

Indeed, $m_T\geq(1-w)rT+w\ m_T=rT+w\ (m_T-rT)$ for $0\leq w\leq 1$ as soon as $m_T\geq rT$ (usual assumption).

2) Second, we have:

$$w^2\left(e^{s_T^2}-1\right)\geq e^{w^2s_T^2}-1.$$

Indeed, introduce the function $\chi(x)=x\left(e^{s_T^2}-1\right)-e^{xs_T^2}+1$ on $[0,1]$. We have $\chi'(x)=\left(e^{s_T^2}-1\right)-s_T^2e^{xs_T^2}$ and $\chi'(x)=0\Longleftrightarrow\left(e^{s_T^2}-1\right)=s_T^2e^{xs_T^2}$. Then, note that $\chi'(x)=\left(e^{s_T^2}-1\right)-s_T^2e^{xs_T^2}$ is decreasing w.r.t. x in $[0,1]$ and $\chi'(0)=\left(e^{s_T^2}-1\right)-s_T^2\geq 0$, $\chi'(1)=-1<0$. Thus, there exist one and only one x^* in $[0,1]$ such that $\chi'(x^*)=0$. Finally, since $\chi(0)=\chi(1)=0$, we deduce that $\chi(x)\geq 0$ on $[0,1]$. By applying previous result to $x=w^2$, we get $w^2\left(e^{s_T^2}-1\right)\geq e^{w^2s_T^2}-1$. Therefore, using (1) and (2), we prove that $Variance\left[R_T^{CM}\right]\leq Variance\left[R_T^{BH}\right]$. ■

- We examine now the computation of skewness and kurtosis.

When there is no jump, we apply the following general results about skewness and excess kurtosis of a Lognormal distribution, namely: if $X=e^Y$ where Y follows a Gaussian distribution $\mathcal{N}\left(m,s^2\right)$ then:

$$Sk\left[X\right]=\sqrt{e^{s^2}-1}\left(2+e^{s^2}\right), \tag{44}$$

$$ExcessKurt\left[X\right]=\left(e^{4s^2}+2e^{3s^2}+3e^{2s^2}-6\right). \tag{45}$$

Thus, the skewness and excess kurtosis are respectively equal to:

$$Sk\left[R_T^{BH}\right]=\sqrt{e^{s_T^2}-1}\left(2+e^{s_T^2}\right), \tag{46}$$

$$Sk\left[R_T^{CM}\right]=\sqrt{e^{w^2s_T^2}-1}\left(2+e^{w^2s_T^2}\right), \tag{47}$$

and

$$ExcessKurt\left[R_T^{BH}\right]=\left(e^{4s_T^2}+2e^{3s_T^2}+3e^{2s_T^2}-6\right), \tag{48}$$

$$ExcessKurt\left[R_T^{CM}\right]=\left(e^{4w^2s_T^2}+2e^{3w^2s_T^2}+3e^{2w^2s_T^2}-6\right). \tag{49}$$

The multidimensional case:

To compare the variances for varying correlations, we note that we have:

$$Variance\left[R_T^{BH}\right] =$$

$$\sum_{i=1}^{d} w_i^2 \exp\left[2\left(\int_0^T \mu_{i,t}dt\right)\right]\left(e^{\int_0^T \sigma_{i,t}^2 dt}-1\right)$$
$$+2\sum_{1\leq i<j\leq d} w_i w_j \exp\left[\int_0^T \left[\mu_{i,t}+\mu_{j,t}\right]dt\right]\left(\exp\left[\int_0^T \sigma_{i,t}\sigma_{j,t}\rho_{i,j}dt\right]-1\right), \tag{50}$$

$$Variance\left[R_T^{CM}\right] =$$

$$\exp\left[2\left((1-\sum_{i=1}^{d} w_i)rT+\sum_{i=1}^{d} w_i\left(\int_0^T \mu_{i,t}dt\right)\right)\right]\left(e^{\sum_{i=1}^{d} w_i^2\left(\int_0^T \sigma_{i,t}^2 dt\right)+2\sum_{1\leq i<j\leq d} w_i w_j \int_0^T \sigma_{i,t}\sigma_{j,t}\rho_{i,j}dt}-1\right). \tag{51}$$

Therefore, when all drifts μ_i are equal with no riskless asset and $d=2$, we have only to compare

$$w^2\left(e^{\int_0^T \sigma_{1,t}^2 dt}-1\right)+(1-w)^2\left(e^{\int_0^T \sigma_{2,t}^2 dt}-1\right)+2w(1-w)\left(\exp\left[\int_0^T \rho\sigma_{1,t}\sigma_{2,t}dt\right]-1\right)$$

with

$$e^{w^2\left(\int_0^T \sigma_{1,t}^2 dt\right)+(1-w)^2\int_0^T \sigma_{2,t}^2 dt+2w(1-w)\rho\int_0^T \sigma_{1,t}\sigma_{2,t}dt}-1.$$

This is equivalent to the comparison of

$$w^2 e^{\int_0^T \sigma_{1,t}^2 dt}+(1-w)^2 e^{\int_0^T \sigma_{2,t}^2 dt}+2w(1-w)\exp\left[\rho\int_0^T \sigma_{1,t}\sigma_{2,t}dt\right]$$

with

$$e^{w^2\left(\int_0^T \sigma_{1,t}^2 dt\right)+(1-w)^2\int_0^T \sigma_{2,t}^2 dt+2w(1-w)\rho\int_0^T \sigma_{1,t}\sigma_{2,t}dt}.$$

Let us denote:

$$h(\rho) =$$

$$w^2 e^{\int_0^T \sigma_{1,t}^2 dt}+(1-w)^2 e^{\int_0^T \sigma_{2,t}^2 dt}+2w(1-w)\exp\left[\rho\int_0^T \sigma_{1,t}\sigma_{2,t}dt\right]$$
$$-$$
$$e^{w^2\left(\int_0^T \sigma_{1,t}^2 dt\right)+(1-w)^2\int_0^T \sigma_{2,t}^2 dt+2w(1-w)\rho\int_0^T \sigma_{1,t}\sigma_{2,t}dt}.$$

We have:

$$h'(\rho) =$$

$$2w(1-w)\int_0^T \sigma_{1,t}\sigma_{2,t}dt \exp\left[\rho\int_0^T \sigma_{1,t}\sigma_{2,t}dt\right]$$
$$-$$
$$2w(1-w)\rho\int_0^T \sigma_{1,t}\sigma_{2,t}dt\ e^{w^2\left(\int_0^T \sigma_{1,t}^2 dt\right)+(1-w)^2\left(\int_0^T \sigma_{2,t}^2 dt\right)+2w(1-w)\rho\int_0^T \sigma_{1,t}\sigma_{2,t}dt}$$

Thus:

$$h'(\rho) = 2w(1-w)\left(\int_0^T \sigma_{1,t}\sigma_{2,t}dt\right)\times$$
$$\left(\exp\left[\rho\int_0^T \sigma_{1,t}\sigma_{2,t}dt\right] - \exp\left[w^2\left(\int_0^T \sigma_{1,t}^2 dt\right) + (1-w)^2\left(\int_0^T \sigma_{2,t}^2 dt\right) + 2w(1-w)\rho\int_0^T \sigma_{1,t}\sigma_{2,t}dt\right]\right).$$

Therefore, we get the following equivalences:

$$h'(\rho) \geq 0 \Longleftrightarrow$$
$$\rho\int_0^T \sigma_{1,t}\sigma_{2,t}dt \geq w^2\left(\int_0^T \sigma_{1,t}^2 dt\right) + (1-w)^2\left(\int_0^T \sigma_{2,t}^2 dt\right) + 2w(1-w)\rho\int_0^T \sigma_{1,t}\sigma_{2,t}dt \Longleftrightarrow$$
$$\rho \geq \rho^* = \frac{w^2\left(\int_0^T \sigma_{1,t}^2 dt\right) + (1-w)^2\left(\int_0^T \sigma_{2,t}^2 dt\right)}{\left(\int_0^T \sigma_{1,t}\sigma_{2,t}dt\right)[1-2w(1-w)]}.$$

Consequently, the difference $Variance\left[R_T^{BH}\right] - Variance\left[R_T^{CM}\right]$ reaches a minimum at ρ^* if condition $\rho^* \leq 1$ is satisfied. Note that condition $\rho^* \leq 1$ is equivalent to:

$$\left(w\sqrt{\int_0^T \sigma_{1,t}^2 dt} + (1-w)\sqrt{\int_0^T \sigma_{2,t}^2 dt}\right)^2 \leq \int_0^T \sigma_{1,t}\sigma_{2,t}dt.$$

The previous condition is not always satisfied. It depends on the choice of the parameters w, $\sigma_{1,t}$ and $\sigma_{2,t}$.

Therefore, we must distinguish two cases:

Case 1: $\rho^* \geq 1$. In that case, the difference $Variance\left[R_T^{BH}\right] - Variance\left[R_T^{CM}\right]$ is decreasing with respect to the correlation ρ.

Case 2: $\rho^* < 1$. In that case, the difference $Variance\left[R_T^{BH}\right] - Variance\left[R_T^{CM}\right]$ is first decreasing then increasing with respect to the correlation ρ. It means that, at $\rho = \rho^*$, the advantage of the constant mix strategy is all the weaker from the point of view of variance. In that case, the minimum value of the difference of the two variances is given by:

$$Variance\left[R_T^{BH}\right] - Variance\left[R_T^{CM}\right] = \exp\left[2\left(\int_0^T \mu_t dt\right)\right] h(\rho^*)$$

with

$$h(\rho^*) = w^2 e^{\int_0^T \sigma_{1,t}^2 dt} + (1-w)^2 e^{\int_0^T \sigma_{2,t}^2 dt} - [1-2w(1-w)] \exp\left[\frac{w^2\left(\int_0^T \sigma_{1,t}^2 dt\right) + (1-w)^2 \left(\int_0^T \sigma_{2,t}^2 dt\right)}{[1-2w(1-w)]}\right].$$

6.4 Proof of intersection of the cumulative distribution functions

- Note that for $x < (1-w)e^{rT}$, $F_{R_T^{BH}}(x) = 0$ while $F_{R_T^{CM}}(x)$ is strictly positive. Thus $F_{R_T^{BH}}(x) - F_{R_T^{CM}}(x)$ is negative.

- For $x > (1-w)e^{rT}$, let us examine the following function:

$$\psi(x) = wLog\left(\frac{x-(1-w)e^{rT}}{w}\right) + \frac{1}{2}\sigma^2 w(1-w)T - \left(Log\,(x) - (1-w)rT\right).$$

We get:

$$\begin{aligned} \psi'(x) &= \frac{w}{x-(1-w)e^{rT}} - \frac{1}{x}, \\ &= \frac{(w-1)\,x + (1-w)e^{rT}}{[x-(1-w)e^{rT}]\,x} \end{aligned}$$

Thus we have:

$$\psi'(x) = 0 \Longleftrightarrow x^* = e^{rT}.$$

Note that:

$$\psi(x^*) = \frac{1}{2}\sigma^2 w(1-w)T.$$

We have also

$$\lim_{x\to+\infty} \psi(x) < 0.$$

Consequently, using intermediate value theorem for continuous functions jointly with strictly monotony of function ψ on both subintervals $[0, x^*]$ and $[x^*, +\infty[$, we deduce that there exist exactly two values of the risky asset return such that the two cdf curves intersect: one on $](1-w)e^{rT}, e^{rT}[$; the other one on $]e^{rT}, +\infty[$.

6.5 Optimal CPPI portfolio for the expected utility criterion

In what follows, we consider d financial assets S_i described from a multidimensional Brownian motion:

$$dS_{i,t} = S_{i,t}(\mu_i dt + \sigma_i dW_{i,t}),$$

where $W_t = (W_{i,t})_{1\le i\le d}$ is a $d-$dimensional Brownian motion with correlation matrix given by: $\Sigma_c = [\rho i, j]_{1\le i,j\le d}$.

Denote:

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & & & 0 \\ 0 & & & 0 \\ 0 & 0 & 0 & \sigma_d \end{bmatrix}.$$

The variance-covariance matrix Σ_S of asset prices S is given by: $\Sigma_S = \left(\sigma_{S_i,S_j}\right)_{1\leq i,j\leq d}$ with

$$\sigma_{S_i,S_j} = S_{i,0}S_{j,0}\exp\left[\left(\mu_i+\mu_j\right)t\right]\left(\exp\left[\left(\sigma_i\sigma_j\rho_{i,j}\right)t\right]-1\right)$$

Indeed, we have:

$$\begin{aligned} S_{i,t} &= S_{i,0}\exp\left[\left(\mu_i - \frac{1}{2}\sigma_i^2\right)t + \sigma_i W_{i,t}\right], \\ S_{j,t} &= S_{j,0}\exp\left[\left(\mu_j - \frac{1}{2}\sigma_j^2\right)t + \sigma_j W_{j,t}\right], \end{aligned}$$

from which we get:

$$S_{i,t}S_{j,t} =$$

$$S_{i,0}S_{j,0}\exp\left[\left(\mu_i - \frac{1}{2}\sigma_i^2 + \mu_j - \frac{1}{2}\sigma_j^2\right)t + \sigma_i W_{i,t} + \sigma_j W_{j,t}\right] =$$

$$S_{i,0}S_{j,0}\exp\left[\left(\mu_i+\mu_j+\sigma_i\sigma_j\rho_{i,j}\right)t - \frac{1}{2}\left(\sigma_i^2+\sigma_j^2+2\sigma_i\sigma_j\rho_{i,j}\right)t + \sigma_i W_{i,t} + \sigma_j W_{j,t}\right]$$

Thus:

$$\mathbb{E}\left[S_{i,t}S_{j,t}\right] = S_{i,0}S_{j,0}\exp\left[\left(\mu_i+\mu_j+\sigma_i\sigma_j\rho_{i,j}\right)t\right].$$

Consequently, we get:

$$\begin{aligned} \sigma_{S_i,S_j} &= Covariance\left(S_{i,t};S_{j,t}\right) = \mathbb{E}\left[S_{i,t}S_{j,t}\right] - \mathbb{E}\left[S_{i,t}\right]\mathbb{E}\left[S_{j,t}\right] \\ &= S_{i,0}S_{j,0}\exp\left[\left(\mu_i+\mu_j\right)t\right]\left(\exp\left[\left(\sigma_i\sigma_j\rho_{i,j}\right)t\right]-1\right) \end{aligned}$$

In what follows, we show how the Brownian motion is function of the risky asset prices. Since the risky asset prices are defined from the relations:

$$dS_{i,t} = S_{i,t}(\mu_i dt + \sigma_i dW_{i,t}),$$

we deduce that:

$$S_{i,t} = S_{i,0}\exp((\mu_i - 1/2\sigma_i^2)t + \sigma_i W_{i,t}).$$

Therefore, we get:

$$\exp(\sigma_i W_{i,t}) = S_{i,t}/S_{i,0}\exp(-(\mu_i - 1/2\sigma_i^2)t),$$

and finally:

$$W_{i,t} = Log\left[S_{i,t}/S_{i,0}\right] - (\frac{\mu_i}{\sigma_i} - 1/2\sigma_i)t).$$

In what follows, we use the following notations (see Jacod and Shiryaev, 2002). For any two semimartingales X and Y, $[X,Y]$ denotes the quadratic variation of the processes X and Y. The process $\langle X,Y\rangle$ denotes the predictable compensator of these processes. Recall that we have:

$$[X,Y]_t = \langle X,Y\rangle_t + \sum_{s\le t} \Delta X_s \Delta Y_s,$$

where ΔX_t and ΔY_t denote the jumps of the processes and $\langle X,Y\rangle_t$ is defined from the following condition: the respective martingales parts M^X and M^Y of X and Y are such that $\left(M_t^X M_t^Y - \langle X,Y\rangle_t\right)_t$ is a (local) martingale.

We have also: (integration by part formula)

$$d\left(XY\right) = XdY + YdX + d\left[X,Y\right].$$

The process $\mathcal{E}(X)$ denotes the Dade-Doléans stochastic exponential, defined from the stochastic differential equation (SDE):

$$d\mathcal{E}(X) = \mathcal{E}(X)dX.$$

Note that, for continuous semimartingales X, we get:

$$\mathcal{E}(X_t) = \mathcal{E}(X_0)\exp\left[X_t - \frac{1}{2}\langle X,X\rangle_t\right].$$

Assuming that Σ_c and Σ are invertible, the financial market is arbitrage-free and complete. The risk-neutral probability $\mathbb{Q}$ exists and is unique. It is defined from its Radon-Nikodym density η with respect to the objective probability $\mathbb{P}$.

Using the martingale representation theorem for the Brownian filtration, this density is associated to d market risk premia, $\lambda_1, ..., \lambda_d$ and is given by:

$$\eta_t = \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t\right] = \mathcal{E}\left[-\sum_{j=1}^{d}\lambda_j W_{j,t}\right],$$

where $\mathcal{E}(.)$ is the Dade-Doléans stochastic exponential and with λ_i satisfying:

$$\Sigma.\Sigma_c.\Lambda = M - r\mathbb{I}. \quad (52)$$

where: $M =^t [\mu_i]_{1\le i\le d}$ and $\Lambda =^t [\lambda_i]_{1\le i\le d}$.

The previous result is established by using the Girsanov's theorem. Each of the d basic assets $S_{i,t}$ must satisfy the following condition: when they are discounted by the nominal money market account C, they must be martingales with respect to the risk-neutral probability $\mathbb{Q}$. This is equivalent to the fact that when they are multiplied by the Radon-Nikodym density η and divided by C, they must be martingales with respect to the historical probability $\mathbb{P}$. Note that we have:

$$S_{i,t}\eta_t/C_t = \mathcal{E}\left[(\mu_i - r)\,t + \sigma_i W_{i,t}\right]\mathcal{E}\left[-\sum_{j=1}^{d}\lambda_j W_{j,t}\right],$$

which is also equal to (Yor's formula):

$$\mathcal{E}\left[(\mu_i - r)\,t + \sigma_i W_{i,t} - \sum_{j=1}^{d}\lambda_j W_{j,t} - \sigma_i\left(\sum_{j=1}^{d}\lambda_j\rho_{i,j}\right)t\right].$$

The fact that the processes $(S_{i,t}\eta_t/C_t)$ are martingales with respect to $\mathbb{P}$ is equivalent to the following property: their bounded variation components are equal to 0. This later condition implies the four following equalities: for all $i = 1, ..., d$,

$$(\mu_i - r) - \sigma_i\left(\sum_{j=1}^{d}\lambda_j\rho_{i,j}\right) = 0,$$

which leads to Equation (52).

Then, we determine the Radon-Nikodym density as function of the risky asset prices as follows:

$$\eta_T = \mathcal{E}\left[-\sum_{i=1}^{d}\lambda_i W_{i,t}\right]$$

$$= \exp\left[-1/2\left(\sum_{i=1}^{d}\lambda_i\lambda_j\rho_{i,j}\right)t - \sum_{i=1}^{d}\lambda_i W_{i,t}\right]$$

$$= \exp\left[-1/2\left(\sum_{i=1}^{d}\lambda_i\lambda_j\rho_{i,j}\right)t\right]\exp\left[-\sum_{i=1}^{d}\left(\frac{\lambda_i}{\sigma_i}\right)\sigma_i W_{i,t}\right]$$

$$= \exp\left[-1/2\left(\sum_{i=1}^{d}\lambda_i\lambda_j\rho_{i,j}\right)t\right]\prod_{i=1}^{d}(S_{i,t}/S_{i,0})^{\left(-\frac{\lambda_i}{\sigma_i}\right)}\exp\left(\frac{\lambda_i}{\sigma_i}(\mu_i - 1/2\sigma_i^2)t\right).$$

Now, we can determine the optimal portfolio for HARA utility function defined by:

$$U(v) = (v - pV_0)^{(1-\gamma)}/(1-\gamma),$$

with $\gamma \neq 1$ and p denotes a guaranteed proportion of the initial investment V_0. This implies that the inverse J of the marginal utility is given by:

$$J(y) = U\prime^{-1}(y) = pV_0 + y^{-1/\gamma}.$$

Using the seminal result of Cox-Huang (1989), we deduce that the optimal portfolio value is equal to:

$$V_T^* = J(a.\eta_T) = pV_0 + c\eta_T^{-1/\gamma},$$

where c is a constant deduced from the budget constraint:

$$V_0 = e^{-rT}\mathbb{E}\left[V_T^*\right] = pV_0 e^{-rT} + ce^{-rT}\mathbb{E}\left[\eta_T^{-1/\gamma}\right].$$

Therefore, the portfolio value V_T^* is a function of the basic assets S_i given by:

$$V_0 + c\exp\left[1/2\frac{1}{\gamma}\left(\sum_{i=1}^{d}\lambda_i\lambda_j\rho_{i,j}\right)T\right]\prod_{i=1}^{d}\left(S_{i,T}/S_{i,0}\right)^{\left(\frac{1}{\gamma}\frac{\lambda_i}{\sigma_i}\right)}\exp\left(-\frac{1}{\gamma}\frac{\lambda_i}{\sigma_i}(\mu_i - 1/2\sigma_i^2)T\right). \tag{53}$$

In what follows, we determine the amounts invested on each respective basic asset S_i. As seen in (53), the portfolio value V_T^* at maturity T is given by:

$$\begin{aligned} V_T^* &= pV_0 + \varphi(T)\prod_{i=1}^{d}\left(S_{i,t}/S_{i,0}\right)^{\left(\frac{1}{\gamma}\frac{\lambda_i}{\sigma_i}\right)} \\ &\text{with} \\ \varphi(T) &= c\exp\left[1/2\frac{1}{\gamma}\left(\sum_{i=1}^{d}\lambda_i\lambda_j\rho_{i,j}\right)t\right]\prod_{i=1}^{d}\exp\left(-\frac{1}{\gamma}\frac{\lambda_i}{\sigma_i}(\mu_i - 1/2\sigma_i^2)t\right) \end{aligned}$$

From the martingale property, the portfolio value V_t at any time t satisfies:

$$V_t^* = e^{-r(T-t)}\mathbb{E}\left[V_T^* \,|\mathcal{F}_t\right].$$

Thus, we have:

$$V_t^* = e^{-r(T-t)}pV_0 + \varphi(T)\prod_{i=1}^{d}\left(S_{i,t}/S_{i,0}\right)^{\left(\frac{1}{\gamma}\frac{\lambda_i}{\sigma_i}\right)}\mathbb{E}\left[\prod_{i=1}^{d}\left(S_{i,T}/S_{i,t}\right)^{\left(\frac{1}{\gamma}\frac{\lambda_i}{\sigma_i}\right)}|\mathcal{F}_t\right].$$

It means that the portfolio value V_t^* is a function of the basic assets $S_{i,t}$ at time t, given by:

$$\begin{aligned} V_t^* &= e^{-r(T-t)}pV_0 + \psi(t)\prod_{i=1}^{d}\left(S_{i,t}/S_{i,0}\right)^{\left(\frac{1}{\gamma}\frac{\lambda_i}{\sigma_i}\right)} \\ &\text{with} \\ \psi(t) &= \varphi(t)\mathbb{E}\left[\prod_{i=1}^{d}\left(S_{i,T}/S_{i,t}\right)^{\left(\frac{1}{\gamma}\frac{\lambda_i}{\sigma_i}\right)}|\mathcal{F}_t\right]. \end{aligned}$$

from which we deduce the result.

Proof. We examine now the term $\mathbb{E}\left[\prod_{i=1}^{d}(S_{i,T}/S_{i,t})^{\left(\frac{1}{\gamma}\frac{\lambda_i}{\sigma_i}\right)}|\mathcal{F}_t\right]$. We have:

$$S_{i,T}/S_{i,t} = \exp\left[m_i^*\left((\mu_i - 1/2\sigma_i^2)(T-t) + \sigma_i\left(W_{i,T} - W_{i,t}\right)\right)\right]$$

$$\begin{aligned} &= \exp\left[m_i^*(\mu_i - 1/2\sigma_i^2)(T-t) + 1/2\left(m_i^*\right)^2\sigma_i^2(T-t)\right] \\ &\quad \times\exp\left[-1/2\left(m_i^*\right)^2\sigma_i^2(T-t) + m_i^*\sigma_i\left(W_{i,T} - W_{i,t}\right)\right]. \end{aligned}$$

Thus:

$$\mathbb{E}\left[\prod_{i=1}^{d}(S_{i,T}/S_{i,t})^{m_i^*}|\mathcal{F}_t\right] = \exp\left[m_i^*(\mu_i - 1/2\sigma_i^2)(T-t) + 1/2\left(m_i^*\right)^2\sigma_i^2(T-t)\right].$$

■

To determine the optimal portfolio shares, first we determine the SDE satisfied by the optimal portfolio value. We have:

$$dV_t^* = dP_t + \psi(t)d\left(\prod_{i=1}^{d}(S_{i,t}/S_{i,0})^{m_i^*}\right) + \prod_{i=1}^{d}(S_{i,t}/S_{i,0})^{m_i^*}\psi'(t)dt$$

We have to identify the factors that multiply the terms $dS_{i,t}/S_{i,t}$. For this purpose, we note that:

$$d\left(\prod_{i=1}^{d}S_{i,t}^{m_i^*}\right) = \sum_{i=1}^{d}\left(m_i^*\left[\prod_{j=1,j\neq i}^{d}S_{j,t}^{m_i^*}\right]S_{i,t}^{m_i^*-1}dS_{i,t}\right) + \vartheta_t,$$

where ϑ is a bounded variation process. Therefore, we have also:

$$d\left(\prod_{i=1}^{d}S_{i,t}^{m_i^*}\right) = \sum_{i=1}^{d}\left(m_i^*\left[\prod_{i=1}^{d}S_{i,t}^{m_i^*}\right]\frac{dS_{i,t}}{S_{i,t}}\right) + \vartheta_t.$$

Since the portfolio value satisfies:

$$V_t^* - P_t = \psi(t)\prod_{i=1}^{d}(S_{i,t}/S_{i,0})^{m_i^*},$$

we deduce that:

$$dV_t^* = \sum_{i=1}^{d}m_i^*\left[V_t^* - P_t\right]\frac{dS_{i,t}}{S_{i,t}} + \varkappa_t,$$

where $\varkappa$ is a bounded variation process. Finally, by identifying the factors of $\frac{dS_{i,t}}{S_{i,t}}$, we conclude that the optimal portfolio shares are given by:

$$e_{i,t}^* = m_i^*\left[V_t^* - P_t\right] = m_i^*C_t.$$

We note that, for the GBM case, the optimal portfolio corresponds to a multidimensional CPPI (see Bertrand and Prigent, 2005, 2011) where amounts respectively invested on the basic assets are proportional to the same cushion with respective constant multiples $m_i^* = \left(\frac{1}{\gamma}\frac{\lambda_i}{\sigma_i}\right)$:

$$\begin{aligned}
V_t^* &= P_t + C_t, \\
&\text{with} \\
P_t &= e^{-r(T-t)} p V_0 \text{ (the floor)}, \\
C_t &= \psi(t) \prod_{i=1}^{d} (S_{i,t}/S_{i,0})^{m_i^*} \text{ (the cushion)}, \\
&\text{and} \\
m_i^* &= \left(\frac{1}{\gamma}\frac{\lambda_i}{\sigma_i}\right).
\end{aligned}$$

This latter formula generalizes the standard one-dimensional CPPI multiple for which we have $m^* = \left(\frac{1}{\gamma}\frac{\mu-r}{\sigma^2}\right)$, since, for $d = 1$, we get exactly this result using the relation $\lambda = \frac{\mu-r}{\sigma}$. Usual financial parameters values yields to $\lambda_i \geq 0$ for all i.

Finally, recall that the constant mix strategy corresponds to $P_t = 0$ with all weigths satisfying:

$$w_i = m_i^* = \left(\frac{1}{\gamma}\frac{\lambda_i}{\sigma_i}\right).$$

Therefore, to get the condition "for all i, $0 \leq w_i \leq 1$" corresponding to the usual constant mix case (i.e. no leverage), the relative risk aversion γ must satisfy:

$$\gamma \geq Max_{i=1,\ldots,d}\left(\frac{\lambda_i}{\sigma_i}\right).$$

6.6 Estimates of the pdf of the two strategies based on US data

In this appendix, we provide additional results about the estimates of the pdf of the two strategies based on US data. We consider two time horizons, namely 60 and 240 months. Results are in accordance with those of Figures 8 and 9 for a time horizon of 120 months.

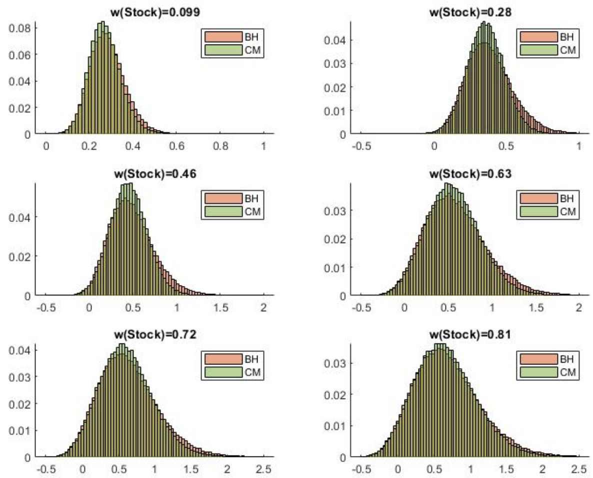


Figure 11: Pdf of CM and BH Portfolios US Stock T-Bill 60 months

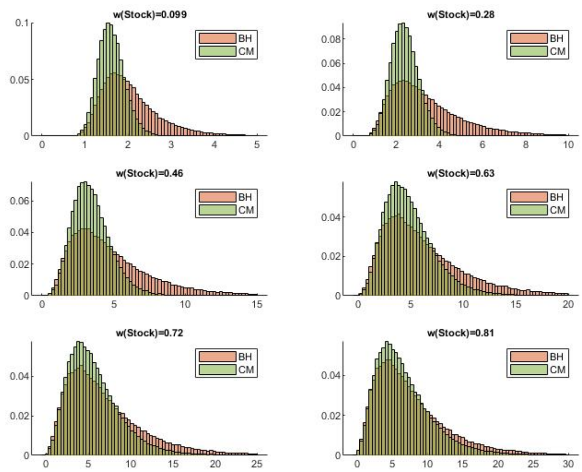


Figure 12: Pdf of CM and BH Portfolios US Stock T-Bill 240 months

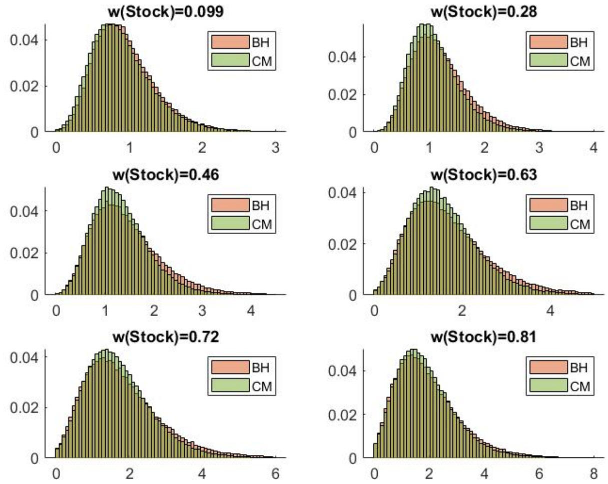


Figure 13: Pdf of CM and BH Portfolios US Stock Bond 60 months

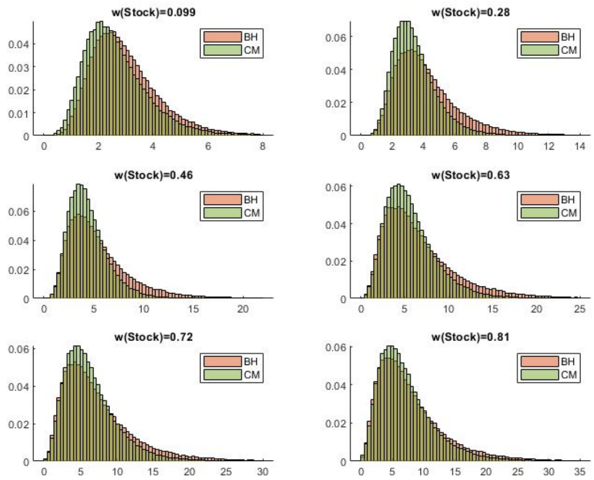


Figure 14: Pdf of CM and BH Portfolios US Stock Bond 240 months

6.7 Compensating variation of the two portfolio strategies in the GBM framework for the CARA case

In what follows, we provide numerical examples of the compensating variations in the standard GBM framework for the CARA case.

Our two numerical base cases are (1) $\mu = 0.06; r = 0.01; \sigma = 0.15; T = 5$, (2) $\mu = 0.12; r = 0.04; \sigma = 0.18; T = 5$.

A CARA utility is defined by $U(x) = -e^{-ax}/a$ where a corresponds to the constant absolute risk aversion.

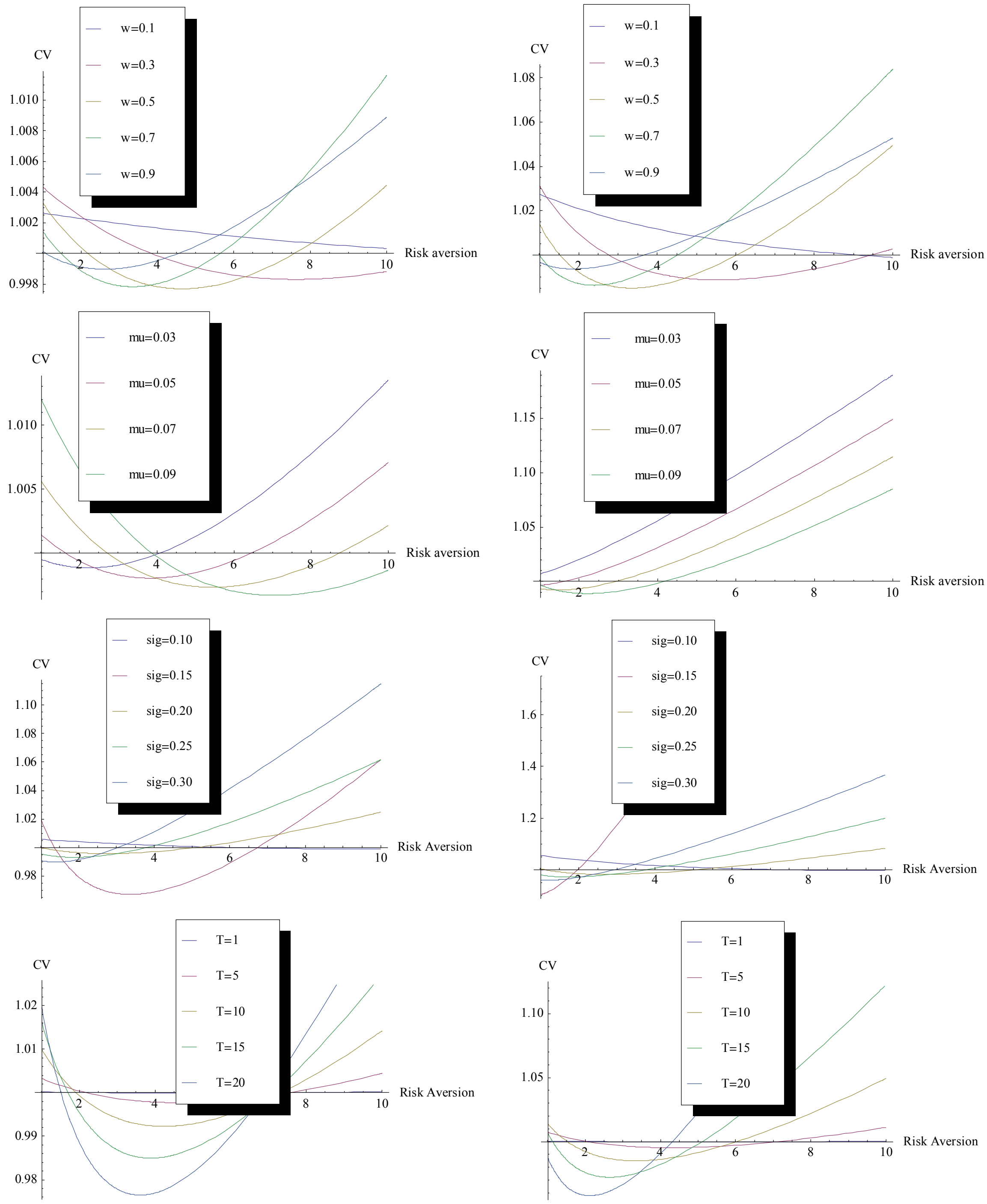
We compute the compensating variation for the CARA case. Using the indifference condition (34), we get:

$$E\left[\exp\left[-aV_0^{BH}\left((1-w)e^{rT} + w\exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma W_T\right]\right)\right]\right] =$$

$$E\left[\exp\left[-aV_0^{CM}\exp\left[\left[(1-w)r + w\mu - \frac{1}{2}w^2\sigma^2\right]T + w\sigma W_T\right]\right]\right].$$

We set $V_0^{BH} = 1$. We search the value of V_0^{CM} for which previous equalitiy holds. When it is higher than 1, it means that the buy-and-hold is preferable, while, when it is smaller than 1, it means the converse. Since we consider here a time horizon equal to 5 years, we can compare the the compensating variation values to implicit management cost applied on this time period. For example, if the compensating variation is equal to 1.10, we can consider that the investor bears an implicit cost of about 2% per year if not having her optimal portfolio weight.

Looking at Figure 15, we can see that the compensating variations are similar to those of the CRRA case (see Figure 7). When the constant mix strategy is preferable (for relatively moderate risk aversion levels), the compensating variation of the buy-and-hold is weak, whereas, when the buy-and-hold, the compensating variation of the constant mix can be very high.



Case 1 Case 2

Figure 15: Compensating variations for the CARA case.

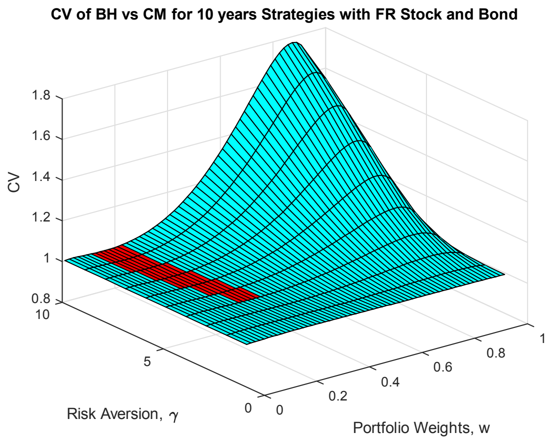
6.8 Compensating variation of the two portfolio strategies for the French case

In what follows we investigate also the French market to examine the robustness of our results for the US market. In Table 10, we summarize the main statistics for the French monthly returns over the whole sample period.

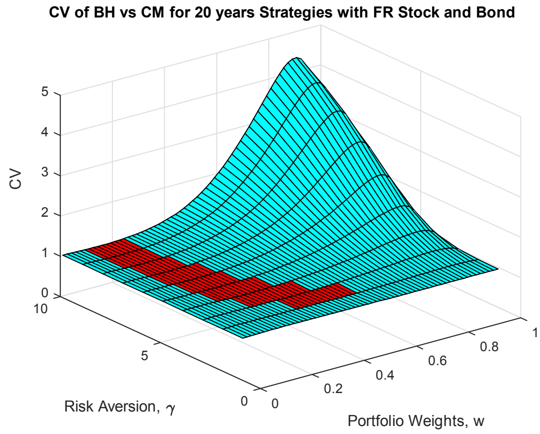
Table 10: Statistics for French Market Returns (1985:03 - 2015:09)

	MSCI TR	**Bond**	**Short Rate**
Mean	0.93%	0.70%	0.39%
Volatility	5.71%	1.82%	0.28%
Skewness	-0.35	0.04	0.43
Kurtosis	3.83	3.35	1.99
Min	-21.82%	-4.23%	0.00%
Max	22.27%	6.76%	1.00%
p-value JB Test	0.25%	33.24%	0.10%

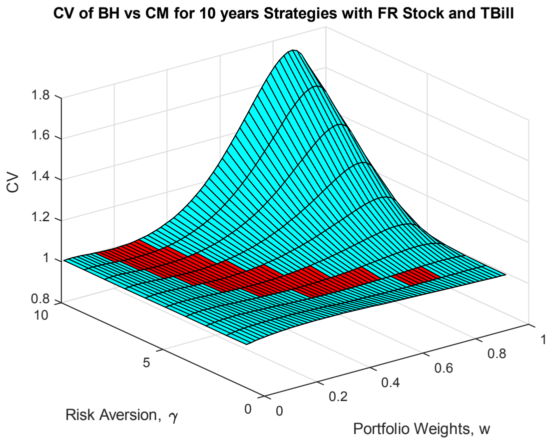
In Figure 10, the CV of the BH versus the CM are displayed for the French market and for portfolios invested in Stock and Bond and in Stock and TBill, both for a ten and a twenty years period on investment. The red area on each figure represents configuration of risk aversion and portfolio weight for which the CM has a higher expected utility, is preferred with respect to the CV criterion. We can see that, for most of the parameter configurations, the BH strategy is preferred to the CM, as for the US case.



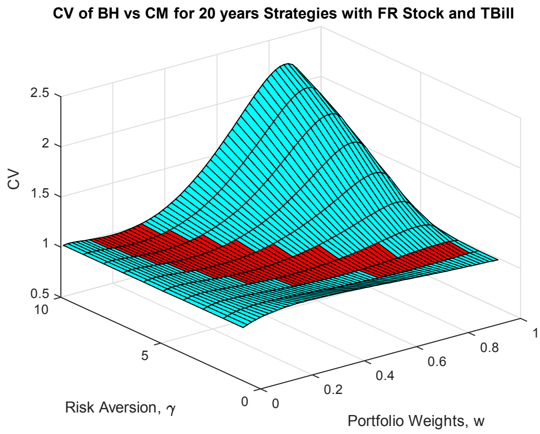
Stock and Bond (10 years)



Stock and Bond (20 years)



Stock and Tbill (10 years)



Stock and Tbill (20 years)

Figure 16: Compensating variations of the two strategies (French case)